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Mathematics for Defense

In response to numerous requests, Professor E. J. McShane, as a member of the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America told the mathematicians present at the recent New Orleans meeting of the Louisiana-Mississippi Section of the Association something of the work of that committee in furthering the defense program. The listeners were impressed with the thorough but unostentatious fashion in which the committee is carrying forward its promotion of appropriate research and with the steps it is taking to see that mathematicians with specialized training can be placed where that training will be utilized when needed. According to Professor McShane, no particularly "showy" results of the work of the committee need be expected.

It seems to us that mathematicians the nation over can observe with profit the spirit and method of the committee in the matter of war preparedness. The real mathematics-for-defense program will not be helped by a ballyhoo type of "Mathematics for Defense" in which vociferation, rather than accomplishment, is most in evidence. Repeated proclamation of the slogan "Mathematics for Defense" alone would have about the same effect upon the public mind as reiterating the statement that "the study of mathematics trains the mind." As a slogan, neither would carry much weight, for instance, in a campaign to restore Mathematics to its former place as a required subject in the public schools.

Aside from the program of research in the advanced branches of mathematics with applications to the science of war and the solution of the specific technical problems of military, naval, and industrial nature, the preparedness work of the teacher of college or secondary school mathematics is clearly the task of providing the strongest training in the fundamentals of mathematics for students in preparation for military and naval service where skill, mathematical knowledge, and judgment are needed. In the words of the War Preparedness Committee, the task would include working to the end that "all competent students in the secondary schools take the maximum amount of mathematics available in their institutions." This means that, as far as the elementary mathematics levels are concerned, the best mathematics for defense is simply *more* mathematics. The answer to the student's question, "Why mathematics for defense?" is closely associated with the answer to his question, "Why mathematics?." The answer to either calls for careful study, if the answer is to be informing and satisfying.

We are not averse to capitalizing on the dramatics of the situation to add to the prestige of mathematics. We simply do not want "Mathematics for Defense" to be classed by the uninformed with the comic strip defense measures, such as Mammy Yocum's pipe in the current adventures of Little Abner.

F. A. RICKEY,
Louisiana State University.

On the Centroid

By N. A. Court
University of Oklahoma

A.

1. Theorem. *If G is the centroid of $(n-1)$ distinct points*

$$A_1, A_2, \cdots, A_{n-1},$$

and A_n any other point, we have

$$\sum_{i=1}^{n-1} A_nA_i^2 = (n-1)A_nG^2 + \tfrac{1}{2}\cdot\frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} A_iA_j^2. \tag{1}$$

To prove this formula by mathematical induction, we assume the validity of the formula

$$\sum_{i=1}^{n-2} A_{n-1}A_i^2 = (n-2)A_{n-1}G'^2 + \tfrac{1}{2}\cdot\frac{1}{n-2}\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} A_iA_j^2, \tag{2}$$

where G' is the centroid of the $(n-2)$ points $A_1, A_2, \cdots, A_{n-2}$, and A_{n-1} any other point.

If in (2) we replace A_{n-1} by A_n we have

$$\sum_{i=1}^{n-2} A_nA_i^2 = (n-2)A_nG'^2 + \tfrac{1}{2}\cdot\frac{1}{n-2}\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} A_iA_j^2. \tag{3}$$

Between the three collinear point A_{n-1}, G, G' we have the relations

$$A_{n-1}G : A_{n-1}G' = (n-2) : (n-1), \quad GG' : A_{n-1}G' = 1 : (n-1), \tag{4}$$

and Stewart's formula applied to the four points A_n, A_{n-1}, G, G' gives, both in magnitude and in sign,

$$A_nA^2_{n-1}\cdot GG' + A_nG^2\cdot G'A_{n-1} + A_nG'^2\cdot A_{n-1}G + GG'\cdot G'A_{n-1}\cdot A_{n-1}G = 0.$$

Replacing in this expression the segments $A_{n-1}G$ and GG' by their values taken from (4) and multiplying the resulting equality by $(n-1)/A_{n-1}G'$ we obtain

$$A_nA^2_{n-1} + (n-2)A_nG'^2 - \frac{n-2}{n-1}A_{n-1}G'^2 = (n-1)A_nG^2.$$

If in this equality we substitute for $(n-2)A_nG'^2$ and $(n-2)A_{n-1}G'^2$ their values taken from (2) and (3), we have

$$A_nA^2_{n-1}+\sum_{i=1}^{n-2} A_nA_i^2-\tfrac{1}{2}\cdot\frac{1}{n-2}\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} A_iA_j^2-\frac{1}{n-1}\sum_{i=1}^{n-2} A_{n-1}A_i^2$$

$$+\tfrac{1}{2}\cdot\frac{1}{(n-1)(n-2)}\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} A_iA_j^2=(n-1)A_nG^2.$$

Combining the first term with the second and the third term with the fifth, this equality may be written as follows

$$\sum_{i=1}^{n-1} A_nA_i^2=\frac{1}{n-1}\sum_{i=1}^{n-2} A_{n-1}A_i^2+\tfrac{1}{2}\cdot\frac{1}{n-1}\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} A_iA_j^2+(n-1)A_nG^2$$

Adding the first two terms on the right hand side we obtain the formula (1).

When $n=3$ the formula (1) gives

$$A_3A_1^2+A_3A_2^2=2A_3G^2+\tfrac{1}{2}\cdot\tfrac{1}{2}(A_1A_2^2+A_2A_1^2),$$

which equality is known to be true from elementary geometry. The proof is thus complete.

It should be observed that in order to prove the formula (1) we made use of Stewart's formula and of the relations (4). Now the latter relations are valid whatever the distribution of the given points may be, and Stewart's formula is valid whether the fourth point considered is or is not collinear with the first three. Consequently the formula (1) is valid for any distribution of the given points.

Also the formula is independent of the dimensionality of the space in which the points are taken.

2. Given n points, the line joining one of these points to the centroid of the remaining $(n-1)$ points may be referred to as a *median* of the n given points. The n points have n medians.

The formula (1) gives the value of the median of the n points issued from the point A_n. It is clear, however, that this formula may be applied to any of the n medians. Thus if we denote by G_k the centroid of the $(n-1)$ points obtained by excluding A_k from the n given points, the median A_kG_k is determined by the formula

$$\sum_{\substack{i=1\\ i\neq k}}^{n} A_kA_i^2=(n-1)A_kG_k^2+\tfrac{1}{2}\cdot\frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n} A_iA_j^2 \tag{5}$$

Now let us make k vary from 1 to n and add the resulting equalities. The left hand side of (5) represents the sum of the squares of the

$(n-1)$ segments obtained by joining the fixed point A_k to the remaining $(n-1)$ given points. If k is made to vary from 1 to n and the corresponding sums added, we shall obtain the sum of the squares of all the $n(n-1)/2$ segments determined by the n given points, each segment counted twice. Thus if we denote by s^2 the sum of the squares of the $n(n-1)/2$ distinct segments determined by the n given points, the sum of the left hand sides of the n equalities considered will be equal to $2s^2$.

The second term on the right hand side of (5) involves $(n-1)(n-2)/2$ segments each counted twice. The sum of this term with the $(n-1)$ analogous terms in the n equalities considered thus involves $n(n-1)(n-2)/2$ segments each counted twice, i. e., it will involve

$$n(n-1)(n-2)/2 : n(n-1)/2 = n-2$$

times all the segments determined by the n given points, each segment counted twice. Thus this sum will be equal to

$$2s^2 \times \tfrac{1}{2} \cdot \frac{1}{n-1} \cdot (n-2) = \frac{n-2}{n-1}\ s^2.$$

We thus have $$2s^2 = (n-1) \sum_{k=1}^{n} A_nA_n^2 + \frac{n-2}{(n-1)}\ s^2, \quad \text{or}$$

$$\sum_{k=1}^{n} A_nA_n^2 = \frac{n}{(n-1)^2}\ s^2. \tag{6}$$

Particular cases of this formula have been noticed.*

3. For the centroid G of the n given points we have

$$A_kG : A_kG_k = (n-1) : n,$$

hence, from (6),

$$\sum_{k=1}^{n} A_kG^2 = s^2/n,$$

i. e., *the sum of the squares of the distances of n given points from their centroid is equal to $1/n$ of the sum of the squares of the $n(n-1)/2$ segments determined by the n given points.*

4. If we replace, in the formula (1), the point A_n by 0, and $(n-1)$ by n, we have

$$\sum_{i=1}^{n} OA_i^2 = nOG^2 + \tfrac{1}{2} \cdot 1/n \sum_{i=1}^{n} \sum_{j=1}^{n} A_iA_j^2.$$

*Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 57, art. 188. The Macmillan Co., 1935.

Now if we assume that O is equidistant from the n points, we have

$$n \cdot OA_i^2 = n \cdot OG^2 + \tfrac{1}{2} \cdot 1/n \cdot 2s^2 \quad \text{(Article 2)}.$$

or

$$OG^2 = OA_i^2 - s^2/n^2. \tag{8}$$

B.

5. Given a tetrahedron $ABCD$ and a point G, let A_0, B_0, C_0, D_0 denote the powers of the points A, B, C, D for the spheres $GBCD$, $GCDA$, $GDAB$, $GABC$, respectively.

Theorem. *If G coincides with the centroid of $ABCD$, we have*

$$A_0 = B_0 = C_0 = D_0, \tag{9}$$

and conversely, if the equalities (9) *hold, the point G coincides with the centroid of $ABCD$.*

Let E, F be the feet of the perpendiculars from the points A, B upon the radical plane CDG of the spheres $GBCD$, $GCDA$, and P, Q their respective centers. We have, both in magnitude and in sign*

$$A_0 = 2EA \cdot PQ, \quad B_0 = 2FB \cdot QP. \tag{10}$$

If U is the trace of AB on the plane CDG, we have, both in magnitude and in sign,

$$EA : FB = UA : UB. \tag{11}$$

Now if G is the centroid of $ABCD$, we have $UA = -UB$, hence, by (11), $EA = -FB$, and therefore, by (10), $A_0 = B_0$. Similarly for any other two quantities involved in (10). Hence the direct proposition.

Conversely, if $A_0 = B_0$, we have, from (10), $EA = -FB$, hence, from (11), $UA = BU$, i. e., the mid-point U of the edge AB lies in the plane GCD containing the point G and the mid-point V of the edge CD. In a similar way it may be shown that the mid-point V lies in the plane GAB vontaining G and U. Consequently the point G is collinear with the mid-points U, V of the two opposite edges AB, CD. Similarly for the other pairs of opposite edges. Hence the converse proposition.

6. Let G_0 be the power of the centroid G of the tetrahedron $ABCD$ with respect to its circumsphere (0). By virtue of the equalities (9) we have†

$$4G_0 = -A_0.$$

**ibid*, p. 184, art. 581.
†*ibid*, p. 187, Art. 594.

On the other hand if in the formula (8) we make $n=4$ and put $OA_i=R$, we have

$$OG^2-R^2=-s^2/16^*$$

Now the left hand side of this formula is equal to G_0, hence

$$A_0=-4G_0=-4(OG^2-R^2)=s^2/4,$$

i. e., *the power of a vertex of a tetrahedron with respect to the sphere determined by the remaining three vertices and the centroid of the tetrahedron is equal to one-fourth of the sum of the squares of the edges of the tetrahedron.*

7. The center P of the sphere $GBCD$ is the point common to the three mediators of the segments GB, GC, GD. Similarly for the centers Q, R, S of the spheres $GCDA$, $GDAB$, $GABC$. Thus the points A, B, C, D are the symmetrics of G with respect to the faces of the tetrahedron $PQRS$. Hence† *the circumcenter O and the centroid G of a tetrahedron $ABCD$ are isogonal conjugate points with respect to the tetrahedron formed by the centers of the spheres $GABC$, $GBCD$, $GCDA$, $GDAB$.*

8. The distances of the centroid G of $ABCD$ from the faces of the tetrahedron $PQRS$ (Article 7) are equal to half the segments GA, GB, GC, GD, i. e., these distances are proportional to the medians of $ABCD$. Now O being the isogonal conjugate of G for $PQRS$ (Article 7), we have:‡ *The distances of the circumcenter of the tetrahedron $ABCD$ from the faces of the tetrahedron $PQRS$ are inversely proportional to the medians of $ABCD$.*

C.

9. Theorem. (a) *If the line joining an arbitrary point U to the centroid G of a triangle ABC meets the sides BC, CA, AB in the points, P, Q, R, we have, both in magnitude and in sign,*

$$\frac{PU}{PG}+\frac{QU}{QG}+\frac{RU}{RG}=3 \quad .$$

(b) *If the line joining an arbitrary point U to the centroid G of a tetrahedron $ABCD$ meets the faces BCD, CDA, DAB, ABC in the points P, Q, R, S, we have, both in magnitude and in sign.*

$$\frac{PU}{PG}+\frac{QU}{QG}+\frac{RU}{RG}+\frac{SU}{SG}=4.$$

*This formula is due to the prolific French writer on the geometry of the triangle and the tetrahedron, V. Thébault. *Nouvelles Annales de Mathématiques*, 1919, p. 425. However, see the same journal for 1845, p. 141.

†*ibid*, p. 244, Art. 750.

‡*ibid*, p. 242, Art. 746.

To prove part (b) denote by A' and K, B' and L, C' and M, D' and N the traces of the lines AG and AU, BG and BU, CG and CU, DG and DU in the faces BCD, CDA, DAB, ABC, respectively. Menelaus' theorem applied to the triangle AGU and the transversal PKA' gives, both in magnitude and in sign,

$$\frac{UP}{PG}\cdot\frac{GA'}{A'A}\cdot\frac{AK}{KU}=-1$$

hence
$$\frac{PU}{PG}=\frac{A'A}{GA'}\cdot\frac{KU}{AK}=\frac{AA'}{GA'}\cdot\frac{UK}{AK}=4\cdot\frac{UK}{AK}.$$

Adding this equality to its three analogues, in the other three faces of the tetrahedron, we obtain

$$\frac{PU}{PG}+\frac{QU}{QG}+\frac{RU}{RG}+\frac{SU}{SG}=4\left(\frac{UK}{AK}+\frac{UL}{BL}+\frac{UM}{CM}+\frac{UN}{DN}\right).$$

Now the sum in the parenthesis is equal to unity,* hence the proposition.

Part (a) is proved in a similar way.

10. Theorem. *If the line drawn through an arbitrary point L of the face ABC of the tetrahedron $DABC$ parallel to the median DD' relative to the face ABC meets the remaining faces DBC, DCA, DAB in the points P, Q, R, we have, both in magnitude and in sign,*

$$LP+LQ+LR=3D'D.$$

The traces X, Y, Z of the lines DP, DQ, DR on the edges BC, CA, AB lie on the line of intersection of the plane ABC with the plane determined by the two parallel lines DD' and $LPQR$.

On the other hand we have, both in magnitude and in sign,

$$\frac{LP}{D'D}=\frac{XL}{XD'},\quad \frac{LQ}{D'D}=\frac{YL}{YD'},\quad \frac{LR}{D'D}=\frac{ZL}{ZD'},$$

hence, adding,

$$LP+LQ+LR=D'D\left(\frac{XL}{XD'}+\frac{YL}{YD'}+\frac{ZL}{ZD'}\right)$$

Now D' being the centroid of the triangle ABC, the sum in the parenthesis is equal to three, by Article 9a, hence the proposition.

**ibid*, p. 115, Art. 339.

11. We have

$$\frac{PU}{PG}=\frac{PG+GU}{PG}=1+\frac{GU}{PG}$$

and similarly for the other ratios of Article 9, hence the formulas of Article 9 may be put in the form*

$$\frac{1}{GP}+\frac{1}{GQ}+\frac{1}{GR}=0 \qquad \frac{1}{GP}+\frac{1}{GQ}+\frac{1}{GR}+\frac{1}{GS}=0$$

i. e., *the sum of the reciprocals of the four (three) directed distances of the centroid of a tetrahedron (triangle) from the points of intersection of the faces (sides) of the tetrahedron (triangle) with a line passing through the centroid, is equal to zero.*

It should be observed that the last formula includes as a special case the property that the centroid of a tetrahedron divides a median in the ratio 1 : 3.

Indeed, if the line $PQRS$ passes through the vertex A, the points P, Q, R coincide with A, and S coincides with the centroid A' of the face BCD, hence the formula considered becomes

$$\frac{3}{GA}+\frac{1}{GA'}=0 \quad \text{or} \quad 3GA'=-GA.$$

Similarly for the median of a triangle.

*ibid. p. 294, ex. 23. *Educational Times*, Vol. 4, series 3 (1918), p. 10, Q. 18390.

"The critical mathematician has abandoned the search for truth. He no longer flatters himself that his propositions are or can be known to him or to any other human being to be true; and he contents himself with aiming at the correct, or the consistent. The distinction is not annulled nor even blurred that consistency contains immanently a kind of truth. He is not absolutely certain, but he believes profoundly that it is possible to find various sets of a few propositions each such that the propositions of each set are compatible, that the propositions of such a set imply other propositions, and that the latter can be deduced from the former with certainty."—From Cassius J. Keyser's ***The Human Worth of Rigorous Thinking***, as published by Scripta Mathematica, Yeshiva College.

The Trisection Problem

By Robert C. Yates
Louisiana State University

CHAPTER III

MECHANICAL TRISECTORS

A variety of mechanisms have been devised for the solution of the Trisection Problem. Some of these mechanisms draw the curves that aid in the solution of the Trisection Equation; others solve the equation directly or are applicable to the immediate division of the angle into three equal parts. It is with the latter sort that we shall concern ourselves in this chapter.

1. *The Graduated Ruler*

Undoubtedly known to Plato and Archimedes was the method of trisecting an angle by means of compasses and *graduated* ruler; that is, one along which marks have been spaced. These marks need not be any specified distance apart and, what is indeed surprising, there need be only *two* marks.

(A) Let us suppose the ruler to have upon one edge* the two points P and R at a distance $2m$ apart. This distance is laid off on one side OB of the given angle. At the midpoint C of this segment, erect the lines perpendicular and parallel to OA. Then slide the ruler through O so that P and R become coincident with these constructed lines as shown in Fig. 13. In that position, the edge of the ruler trisects angle AOB. For, if M is the midpoint of PR,

$$PM = MR = MC = OC = m$$

so that, if $\angle AOR = \theta$, then $\angle MRC = \theta$, the alternate angle formed by the transversal OR and the parallels OA and CR. Since triangles CMR and OCM are isosceles,

$$\angle MCR = \theta, \quad \angle OMC = 2\theta = \angle COM$$

and OR is the trisecting line.

*The other edge is not be used in constructions. A ruler with two straight edges is alone sufficient to make all constructions that are possible by the compasses and simple straightedge [51].

This is the third in a series of five chapters. The next will appear in an early issue.

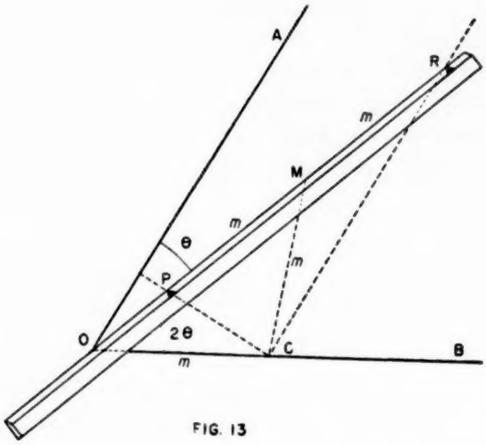

FIG. 13

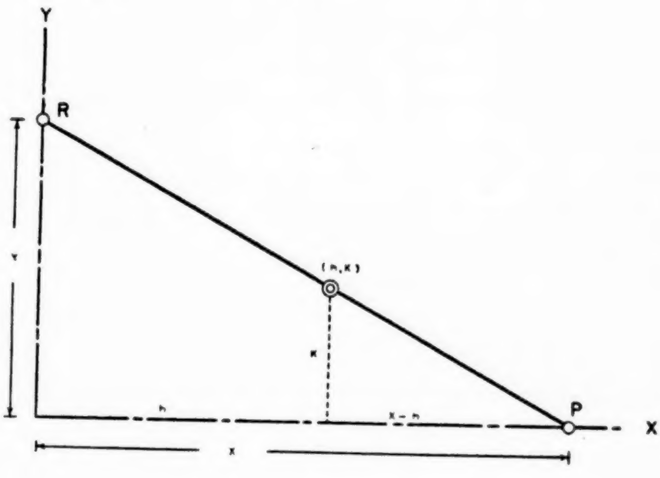

FIG. 14

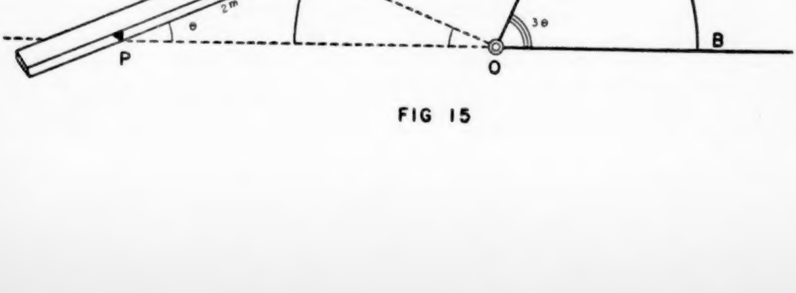

FIG 15

Let us look at the algebraic statement of this "sliding" process. Essentially, it is required that a given segment PR shall be so "inserted" between two fixed perpendicular lines that, extended if necessary, it shall pass through a fixed point O. Let the X and Y axes, Fig. 14, represent the perpendicular lines and let (h,k) be the coordinates of the fixed point. If x and y represent the intercepts of the segment PR, we must have:

$$x^2+y^2=4m^2$$

for the constant length $PR=2m$, and from similar triangles:

$$y/k=x/(x-h).$$

These two equations are now solved for x in terms of the given quantities h, k, and m. From the second $y=kx/(x-h)$ which, substituted in the first, gives:

$$x^2+k^2x^2/(x-h)^2=4m^2,$$

or

$$x^4-2hx^3+(h^2+k^2-4m^2)x^2+8hm^2x-4h^2m^2=0.$$

Thus this insertion problem of Archimedes is one of the fourth degree; that is, there may be four possible positions of the segment. It is evident then that the possession of two marks upon the straightedge, although apparently innocent enough, forms a very powerful tool when used in the insertion manner.

It will be noticed that the insertion principle is fundamental to many of the devices explained in the following paragraphs.

(B) Another mode of solution by the graduated ruler follows directly from Fig. 1 of Chapter I. Construct at the vertex of the given angle AOB the circle with a radius equal to the distance between the marks on the ruler. That is,

$$OR=OA=PR=2m.$$

The points P and R of the ruler are brought into coincidence with the line OB (produced) and the circle, respectively, while the edge of the ruler slides through the point A. Since triangles PRO and ROA, Fig. 15, are isosceles,

$$\angle APO=\angle AOB/3.$$

2. The Compasses of Hermes

Exactly the same idea forms the basis of the compasses with three feet devised by H. Hermes in 1883 [**5**][**17**]. Here two points P and R, Fig. 16, attached to one leg of the compasses at a constant distance

apart, are always in line with A, the point of the other leg. The circle with PR as radius is drawn about the vertex of the given angle. The point A of the compasses is applied to A on the side of the angle as shown and the compasses opened until P and R fall on the line OB and the circle, respectively. Then $\angle APB = \angle AOB/3$.

3. *A Three Bar Apparatus*

Under the insertion method falls the very simple arrangement of three bars shown in Fig. 17. Aubry [**3**] gives credit for this to Ceva but no doubt Pascal also used the instrument to draw his Limaçon. The bars OE and OF are taken equal in length and jointed together at O. The point E is attached so that $CE = OE$ and F is made to slide in a groove along CD. For trisection, the point O is placed at the vertex of the given angle AOB and OF coincident with OB. When C is brought to the produced line OA then

$$\angle ACD = \angle AOB/3.$$

It should be noticed that if CD is fixed, any point of OF traces out an Ellipse. To draw the Limaçon, fix the bar OF and mark the path described by C. If the point C is fixed and O be made to move along a fixed straight line CA, then F describes the Cycloidum anomalarum of Ceva which was discussed in Chapter II.

4. *Ceva's Pantograph.*

Similar to the foregoing is the apparatus of Ceva, [**9**], which was considerably elaborated by Lagarrique in 1831 [**32**]. It is composed of four jointed bars forming a parallelogram with equal sides, two of which are extended. Its application to the angle AOB is shown in Fig. 18. The point O is placed at the vertex of AOB and P is made to move along the bisector of AOB until the extended sides of the parallelogram pass through A and B, the points where the circle of radius $OR = RP$ meets the sides of the given angle. Then, since arcs RS, AS', and BR', being subtended between parallel chords of the circle, are equal, $\angle ROS = \angle AOB/3$.

The instrument can be used as a Pantograph by fixing R to the plane and selecting a point T on SB as the tracer. Then the point T' on OS which is collinear with R and T describes a reduction of the path of T. Compare this device with the one shown in Fig. 17.

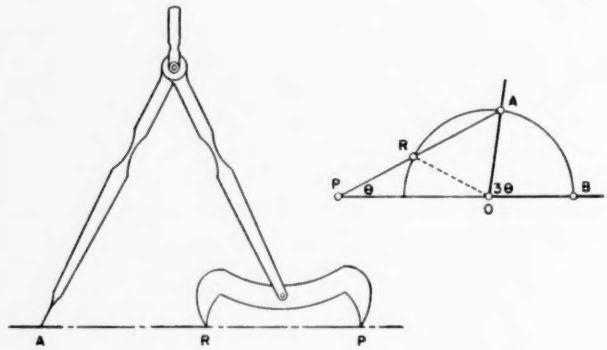

FIG. 16

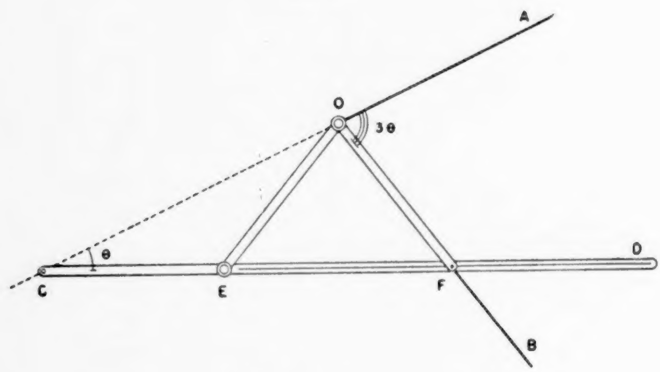

FIG 17

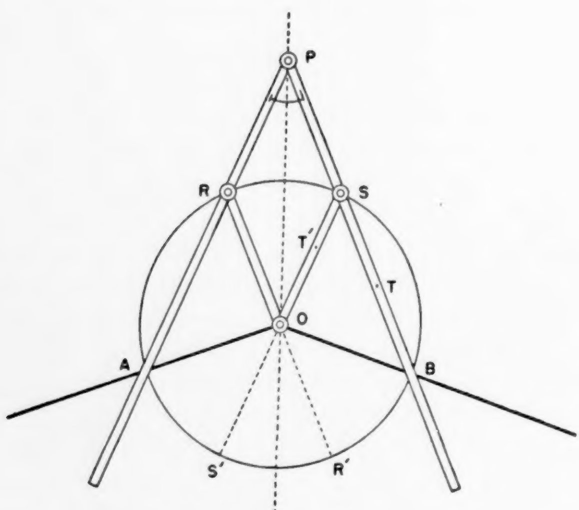

FIG 18

5. *Amadori's Instrument*

Again the same principle is involved in the apparatus of Amadori [**2**]. As indicated in Fig. 19, the straightedge is attached to a plate out of which are cut parts of the circle. The point P of the straightedge moves in a slot along the bisector of the given angle AOB while the other point R moves along the diameter of the circle, this diameter, of course, being equal to the distance PR. When the edge passes through C then the point M determines the trisecting line MOT.

The mechanisms of the preceding paragraphs all contained as the fundamental principle the ability to insert a given segment either between two lines or between a line and a circle in such a way that the line upon which the segment lies passes through some fixed point. The following two very novel and ingenious devices employ the insertion idea but with the different requirement that a line fixed at right angles to the segment shall pass through a given fixed point.

6. *The Carpenter's Square*

A right-angled square with parallel eges whose legs have the same width m, Fig. 20, is first used to draw the line $O'B'$ parallel to OB. It is then placed so that its inner edge passes through O with the corners P and R on OA and $O'B'$ respectively so that $PR=2m$. In this position, it is readily seen that the right triangles, OPT, ORT, and ORK are all congruent with equal angles at O. Then OT and OR are trisecting lines of angle AOB [**44**]. The square was used by Nicholson, [**37**], to draw a trisecting curve and a little later by Aubry [**3**] in this more direct fashion [**51**].

7. *The Tomahawk*

This implement is, in a sense, an improvement over the carpenter's square since it is directly applicable to the trisection of a given angle. Furthermore, the edges of the "handle" need not be parallel to each other. TBP is a semicircle, Fig. 21, with OT a tangent. The point C is the center of the semicircle and A is taken on PT extended so that AT is equal to the radius of the circle. As in Paragraph 6, OT and OC are trisecting lines.

Although the inventor of the Tomahawk is not known, Bergery describes the instrument in the 3rd edition of *Geométrie appliquée a l'industrie*, Metz, 1835. See also [**22**].

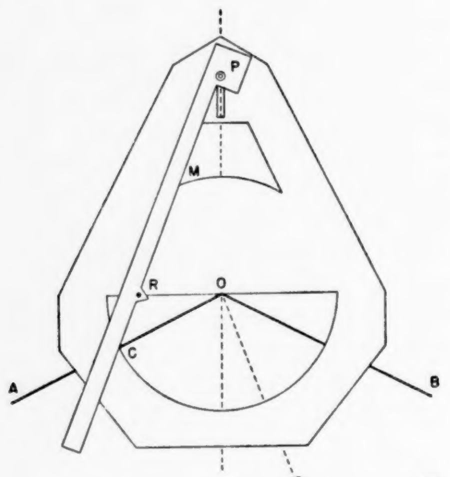

FIG. 19

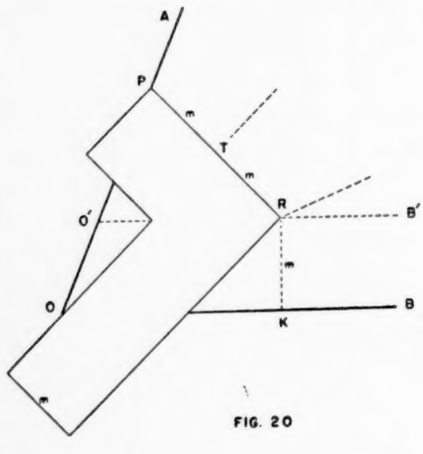

FIG. 20

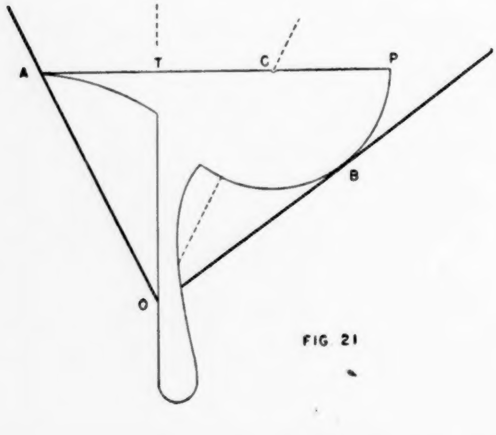

FIG 21

8. *Laisant's Compasses*

A mechanism composed of four straight bars hinged together at one point and forced to make equal angles with each other was given by Laisant [**33**] in 1875. The lengths are chosen so that, Fig. 22,

$$OB = OC, \quad CS' = BS', \quad OD = OA, \quad AS = DS,$$

with S and S' as joints permitted to slide in straight grooves along the two trisecting bars. The triangles OBS', ODS', and OAS are congruent with equal angles at O. The bars OS' and OS are extended beyond O so that the third part can be set off upon the same arc.

9. *Laisant's Mechanism*

Somewhat different is a second device offered by Laisant, ibid. See also [**7**]. $OBCD$ and $CDES$, Fig. 23, are jointed parallelograms with all sides equal. The joint S is forced to move in a straight groove along the rod OD extended. Triangles CBO and CDO are congruent with equal angles at O. Moreover, the same is true of triangles SCO and SEO. Thus

$$\angle BOC = \angle COS = \angle SOE = \angle AOB/3.$$

A glance at the three bars $ODES$ will indicate the connection between this instrument and that shown in Fig. 17, Paragraph 3. Notice that if the bar OB is held fixed, the point E will describe a Limaçon.

10. *Kempe's Trisector*

One of the cleverest amateur mathematicians of the past century was A. B. Kempe who, in 1875, was a young London barrister specializing in ecclesiastical law. To him is due the following elegant mechanism for direct trisection [**29**].

Consider the jointed *crossed parallelogram* $OC'O'C$, Fig. 24, composed of four bars equal in pairs. That is,

$$OC = O'C' = b \quad \text{and} \quad OC' = O'C = c.$$

It is evident that no matter how the mechanism is deformed,

$$\angle COC' = \angle CO'C' = \theta.$$

Now let us attach two more bars OD and DE as shown in Fig. 25 so that

$$DE = OC = b \quad \text{and} \quad OD = CE = d.$$

With this, $ODEC$ itself is another smaller crossed parallelogram with

$$\angle DOC = \angle CED.$$

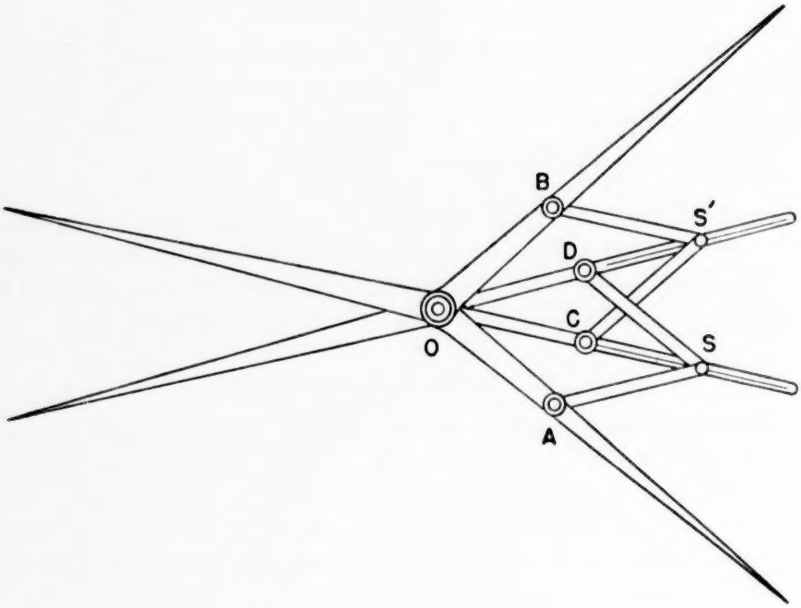

FIG 22

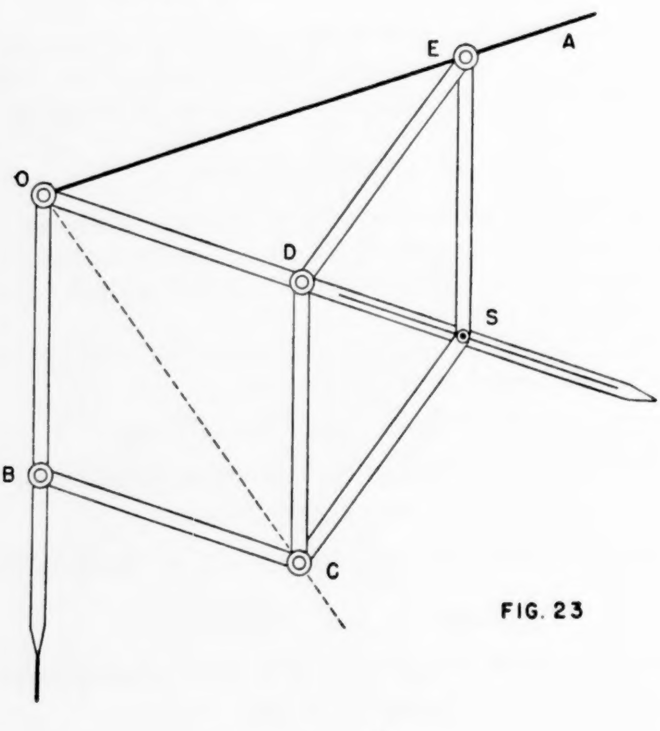

FIG. 23

Let us see if it is possible to arrange matters so that $\angle DOC = \angle COC' = \theta$ throughout all deformations of the second mechanism. If these angles are to be equal, the two crossed parallelograms are always similar

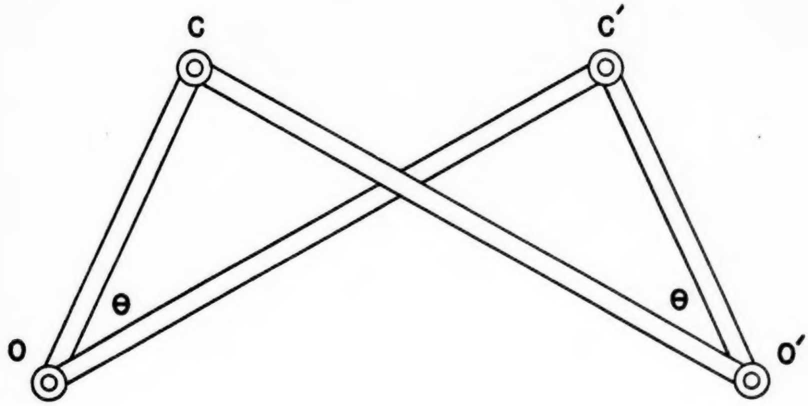

FIG. 24

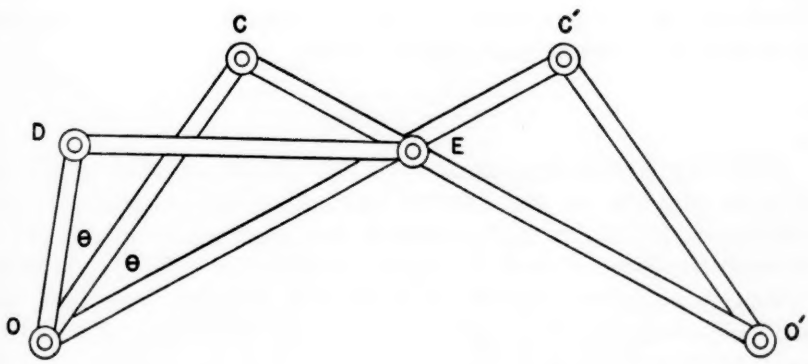

FIG. 25

since they already have equal angles at D, C, and C'. Consequently, we must have the proportion:

$$OD/OC = OC/OC' \quad \text{or} \quad d/b = b/c,$$

or

$$b^2 = cd.$$

This means that the length of OC (and of $O'C'$) must be a mean proportional between the lengths OC' and OD; for example, $d=1$, $b=2$, $c=4$.

From this discussion, it is obvious that two more bars, OF and FG, may be attached in the same fashion to give *three* equal angles

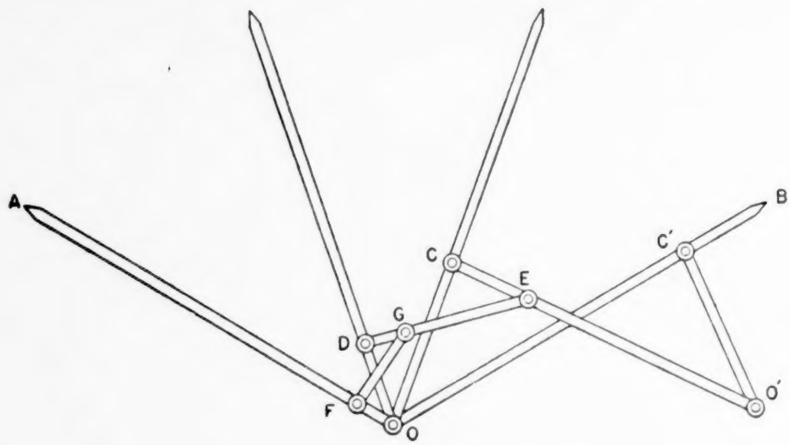

FIG 26

at O, thus producing the Kempe trisector shown in Fig. 26. For the construction of the mechanism, it will be found convenient to take 1, 2, 4, and 16 inches as appropriate lengths.

11. *A Linkage*

With the Kempe arrangement of two crossed parallelograms, we may now improve on the Laisant mechanism of Paragraph 9 [**51**]. Returning to Fig. 23, it will be noticed that the purpose of the slide S is to keep the bars CD and ED equally inclined to OD. Accordingly, by properly attaching crossed parallelograms to these three bars, this equal inclination may be accomplished and the useless bars CS and ES removed without otherwise altering the effect of the mechanism. We have then Fig. 27, a linkage free from the slide and groove combination that is mechanically undesirable.

12. *Sylvester's Isoklinostat*

Another linkage trisector was announced by Sylvester [**47**] in 1875 under the title: "A Lady's Fan". The jointed bars in Fig. 28 are selected so that bar

$$\text{bar } BC = \text{bar } BA + \text{bar } AC$$

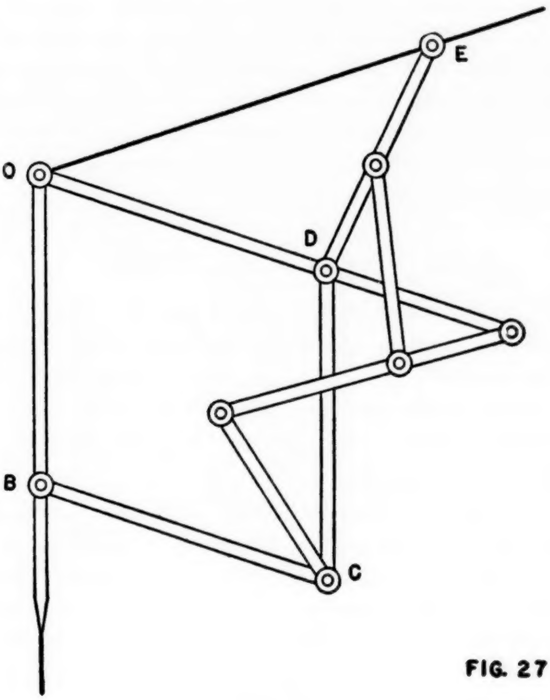

FIG. 27

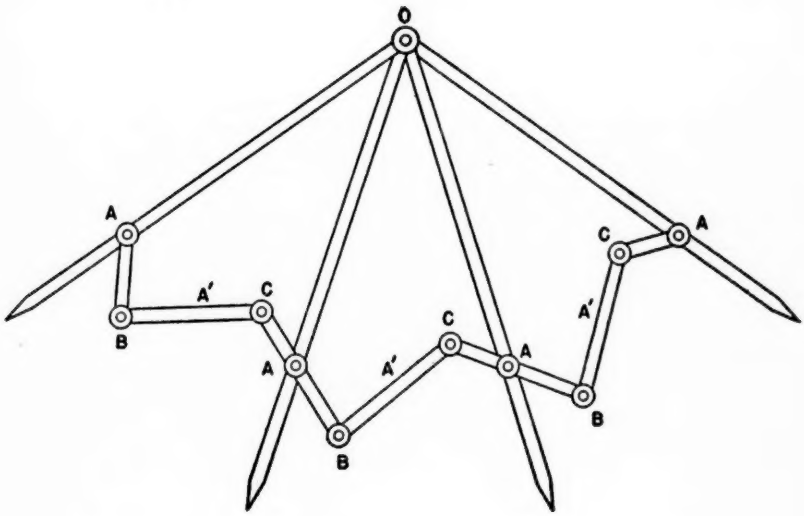

FIG. 28

and the shorter bars are attached to the longer ones at points equidistant from O. By marking the points A' on the cross arms that correspond to the points A it becomes apparent that the quadrilaterals $OABA'$ are all congruent to each other. The same is true for the set $A'CAO$ as well. Thus the long bars always make equal angles at O.

Shortly after its appearance this mechanism was utilized in an optical apparatus to keep moving prisms equally inclined to each other.

13. A Line Motion Trisector

Consider the figures of Paragraphs 9 and 11. In Fig. 23, the point S was constrained to move along the diagonal of the rhombus $CDES$. Let us put the bars ES and CS back into place in Fig. 27. It is clear then that S would move always in line with bar OD. Consequently, *if the bar OD were held fixed as shown in Fig. 29, then S would move in the straight line determined by this bar* [29].

Let the side of the rhombus $OCSE$ be the unit length. By fixing the diagonal OD in a horizontal position, E will move on the unit circle about O, and S along its horizontal diameter. This, it will be noticed, is precisely the underlying principle of the insertion method explained in Paragraph 1. Thus if the bar SE be extended as shown here we need only move this bar so that it will pass through A in order to trisect angle AOB. The angle OSE will, of course, be $\angle AOB/3$.

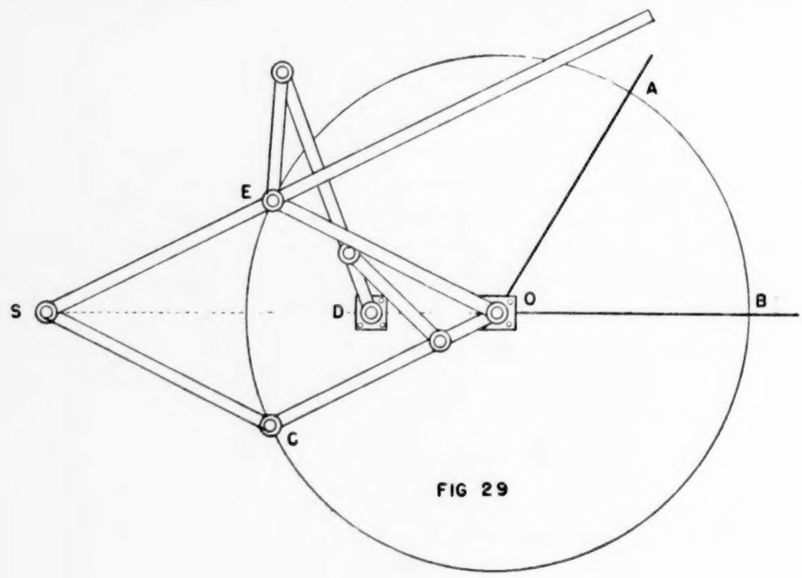

FIG 29

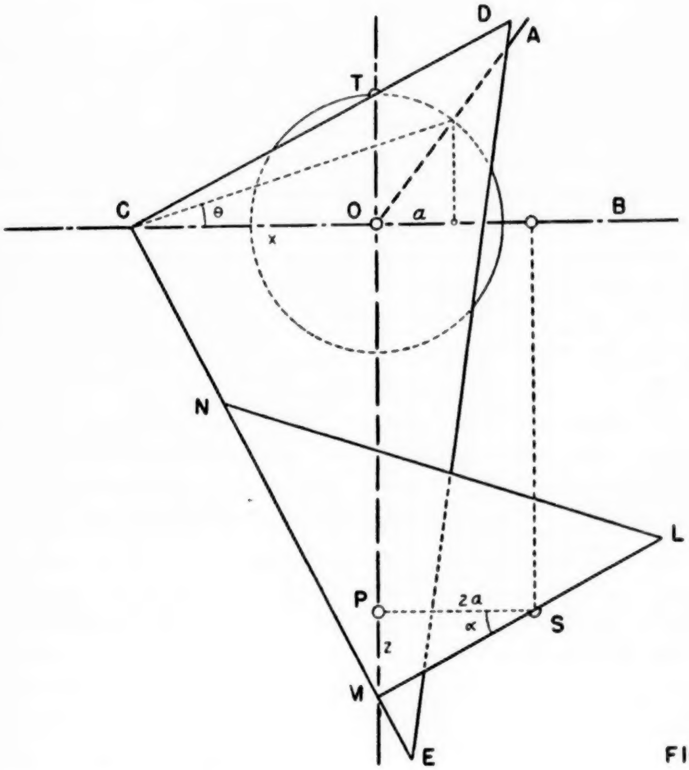

FIG. 30

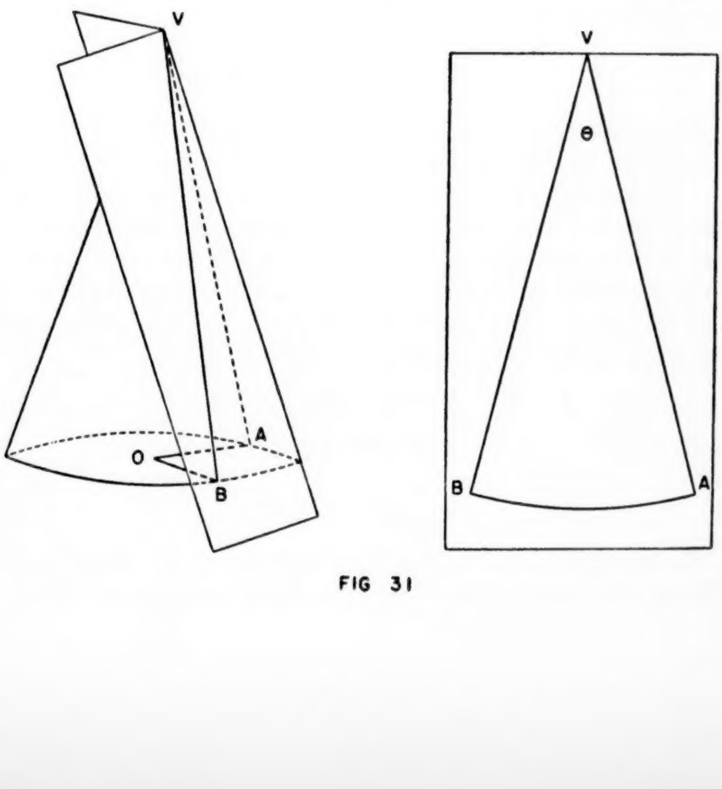

FIG 31

14. *Draughting Triangles*

Two celluloid triangles, each having a right angle, are sufficient equipment to trisect any angle [1][17]. Let $AOB=3\theta$ be the given angle about which is drawn the unit circle, Fig. 30. As usual, a will denote the projection of one unit side upon the other, that is, $a=\cos 3\theta$. Draw the two perpendicular diameters of the circle and mark off two units on the vertical one below the circle to P. At P lay off the distance $2a$ horizontally to S. Slide the two triangles along with their two edges together until the other perpendicular edges pass through S and T. At the same time, the corresponding right angle vertices should lie on the vertical and horizontal lines, respectively. In this position, the line AC determines the angle $\theta=\angle AOB/3$. Let x represent the distance CO and z the distance MP. Since CD and LM are parallel, they make equal angles with the horizontal and thus $\angle OCT=\angle PSM=\alpha$. Furthermore, $\angle CMO=\alpha$. Therefore, the right triangles COT, MOC, and SPM are similar and we have the proportion:

$$1/x=x/(3+z)=z/2a.$$

From the first and last members here: $z=2a/x$ which, substituted in the first two members, gives:

$$1/x=x^2/(3x+2a),$$

or

$$x^3-3x-2a=0,$$

the Trisection Equation.

15. *The Cone Trisector*

The following trisection, given by Aubry [3] is included here for its novelty. A right circular cone, Fig. 31, is constructed having its slant height equal to three times the base radius. The cone is placed so that the center of the base is coincident with the vertex of the given angle, $AOB=3\theta$. Then arc $AB=3r\theta$. A sheet of paper is now wrapped around the cone and the points A, B, and V are marked on it as shown. When the sheet is removed and flattened out, the angle AVB is one-third angle AOB. For, since $AV=3r$,

$$\text{arc } AB=3r(\angle AVB),$$

and thus

$$\angle AVB=\theta=\angle AOB/3.$$

16. *Remarks*

It has been said that Plato objected to all constructions which involved the use of any mathematical instrument other than the

straightedge and compasses. Yet there is every indication that he himself proposed for the solution of the cubic a mechanical arrangement very similar to the pair of right triangles of Paragraph 14. Some historians credit him with the following statement concerning the practice of mechanical solutions: "The good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God." Of course, it is and was realized that the actual drawings made by these instruments were but crude physical representations of the ideals in mind—the "eternal and incorporeal images of thought." But there is nothing un-mathematical about the use of a graduated ruler or any other instrument capable of making appropriate illustrations and physical applications of theory.

By adopting tools other than the classical ones and by altering Platonian rules many intensely interesting and important contributions have been made to the whole field of mathematics. Mascheroni, for example, performed an amazing feat when he proved that the whole of the plane geometry of Euclid could be reconstructed by throwing the straightedge into discard and using only the compasses. If such a possibility had occurred to the Ancients they certainly would not have labeled the straightedge as an instrument of the gods. It too would have been banned and shunned as a degraded tool fit only for the baser uses of the mechanic.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

A History of American Mathematical Journals

By Benjamin F. Finkel
Drury College

(Continued from February, 1941, issue)

The
MATHEMATICAL MONTHLY
Edited by
J. D. Runkle, A. M.,
BACHELOR IN SCIENCE, FELLOW OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES, AND ASSISTANT UPON THE AMERICAN EPHEMERIS AND NAUTICAL ALMANAC.
Vol. I.

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Published by John Bartlett.
London:
TRÜBNER & CO.
1859.

The MATHEMATICAL MONTHLY

It would seem that after more than a half century of experimentation in Mathematical Journalism in the United States, the time for the publication of a mathematical periodical which should become a permanent factor in the development of mathematics in America had arrived. And thus it would appear, considering the favorable auspices under which the *Mathematical Monthly* was established in 1858. The Editor took every precaution to ascertain if the time was opportune

for the publication of such a journal. To this end, early in the year 1858, Mr. Runkle sent out a circular letter setting forth his idea of the character such a journal should possess in order to insure its permanence and greatest usefulness, and asking such questions (four in number) as would elicit replies from those to whom the letter was sent, the nature of which would enable him to form a proper estimate of the probable support the enterprise would receive. This circular letter was sent to nearly all of the most eminent mathematicians and educators in the United States. In a short time, 180 replies were received, to which were added 54 more a little later. To the question, "Do you think there is a present need of a mathematical journal of any kind?" the replies were unanimous in the affirmative. To the question, "Do your views coincide with those here expressed as to its character?" The replies were also unanimous in the affirmative. The replies to the question, "Are you willing to assist in establishing and sustaining a journal by contributing to its pages?" promised, with few exceptions, such aid as was consistent with other duties, while 62 pledged constant and active cooperation.

Moreover, the project came up before Teachers' Associations for discussion, and commendatory resolutions were passed by the New Hampshire State Teachers' Association, the Iowa State Teachers' Association, the New York State Teachers' Association, etc. At the meeting of the American Association for the Advancement of Science held in Baltimore, April 30th, 1858, the committee of the Section of Mathematics and Physics to whom Mr. Runkle's project had been referred, recommended the adoption of a favorable resolution. This committee consisted of A. Casswell, Benjamin Pierce, and Geo. W. Coaklay.

The aim and scope of this journal may best be understood from the Editor's "Introductory Note" which is as follows:

The attempt to establish a Mathematical Journal is a step of too great importance to be taken without due deliberation,—without carefully considering the end to be attained, and the means to be employed in securing it. This end may be either the *advancement* of science or the *elevation* of the standard, of mathematical learning. Now it is not probable that a journal of a high scientific character, having the former end solely in view, could be sustained, for it would contain only one element of interest, and that one for the few professed mathematicians.

But a journal having the latter end in view, if successful in the highest sense, must necessarily to a greater or less degree involve the

former; and the question arises, May not such a journal have a scope sufficiently comprehensive and elastic to embrace all grades of talent and attainment, and, therefore, corresponding elements of interests? If so, then it should embrace students in one extreme, and professed mathematicians in the other; which extremes necessarily include all intermediate grades of teachers and laborers in this vast field.

Should it be a journal merely of problems and solutions? We think not. It should cover the *whole ground* sketched in the following circular note:

To the student, or younger mathematician, problems are indispensable to make him *sure* of, and *ground* him in, the theory, as well as exercise and develop his skill. But it is a serious question, whether it is not a waste of time and energy to put scores of able mathematicians upon the same problem any one of whom has the knowledge and the skill necessary for the complete solution. The problem should, therefore, usually be selected with reference to the learner and so graduated as to suit different degrees of knowledge and skill even in the same branch.

Problems of the highest grade, especially if they are likely to lead the investigator into a comparatively new field, or develop methods, or important practical results, may occasionally be published as challenges; but generally, we think it advisable to publish the solutions of such problems at once, and if those particularly interested in the solution should be led to any new and curious developments, insert them afterwards.

The journal should contain, for all learners, clear and concise notes upon all points of the theory and application in all branches of the science; and these notes should come from able contributors, who can be plain without being weak,—who can unite simplicity of treatment with elegance of style.

It should contain "all scraps of mathematical writing too good to be lost," whether *elementary* or *profound*, whether *original* in *manner* or *matter*, whether complete in themselves or to be resumed at the convenience of the author, whether notices or reviews of matter old or new; in short, everything fitly designated by "notes and queries." Besides, it should contain "carefully elaborated essays, chiefly valuable perhaps as promises of better things hereafter," as well as those of higher character.

There is another large field in which the journal will find its legitimate work,—one in which it can do the double duty of inducting

students and younger mathematicians into the highest departments of the science, and of opening to the able and more experienced an opportunity to contribute their share to the noble work of elevating the student of mathematical learning in the country. All mathematicians know that there are many subjects in the higher departments of the science upon which little, if any thing, has been written among us. Now, if they will take these subjects and develop them fully and systematically through the pages of the journal, they may afterwards be issued in a separate form from the stereotype plates, at a very small cost. In such cases the right and benefit thereof shall vest in the author.

In this way we shall secure the cooperation of all: of students and younger mathematicians, for the range embraces them with their respective abilities and attainments, and, therefore, interests; of professed mathematicians, for, besides the large field specified above, neither their dignity nor scientific character can be affected by communicating notes which might not be of sufficient importance to warrant insertion in a journal of high scientific pretensions.

In fine, then, the journal will be to the professed mathematician a *recreation* and a *study*, while to the student it will be a *study* and an *example*.

Being convinced that a journal of this character, in which all interests shall blend and cooperate, is needed, that it "will occupy ground unoccupied by other periodicals, and will be of great importance in advancing the intellectual character of our country:" and, believing that, if properly sustained by those who ought to have its success most deeply at heart, it can be sustained in its financial department, we have taken the liberty during the last few weeks, as a preliminary inquiry, to send copies of the following note. The replies already received are of such a character as to fully warrant its issue in this more general manner. Every one feeling any interest in this enterprise, is earnestly solicited to express his views fully, and state to what extent, if any, he will cooperate; so that his name may be included in the following list. As this introductory note will be substantially retained in the first number, the names of many of those who will give to the journal its vitality and usefulness will thus be enrolled together: an array and diversity of talent that make it a means of culture, which the friends of good learning in our country "will not willingly let die." The following is the "Circular Note":

NAUTICAL ALMANAC OFFICE,
CAMBRIDGE, MASSACHUSETTS,
February 13, 1858.

DEAR SIR:

Allow me to call your attention to the following considerations: You are aware, that while, almost every science as well as art, has its own appropriate journal, around

which corresponding interests and tastes cluster, by which special research is engaged, and through which all the valuable results are communicated to the world the science of mathematics is without its own particular organ.

Now it seems to us that such a journal is needed; one that shall embrace, among its contributors, the best talent, in order that young laborers in the same field may always have before them a high standard of excellence, and that it may be a fair index of the mathematical ability of the country. On the other hand, however, care should be taken not to graduate it, as a whole, too high above the average attainments of mathematical students; otherwise, only the few would be interested in it or benefitted by it. It should therefore embrace in its pages solutions, demonstrations, and discussions in all branches of the sciences, as well as in all its various applications.

It should contain notes and queries, notices and reviews of all the principal mathematical works issued in this country, as well as in Europe.

In short, it should be the medium of all kinds of information pertaining to the science, to which a large proportion of our mathematical students have at present no ready access. Such is, in brief, our idea of the character the journal should possess to insure to it the greatest usefulness and most permanent success.

This note, with the following queries, will be addressed to many of the most eminent mathematicians and educators in the various parts of the United States; and upon the several replies we shall base our future action.

Do you think there is a present need of a mathematical journal of any kind?

Do your views coincide with those here expressed as to its character? If not, be pleased to state your views.

Are you willing to assist in establishing and sustaining a journal by contributing to its pages?

Will you allow such use to be made of your reply to this note as may be proper to carry the proposed plan into effect?

A *decided* reply is respectfully solicited, whether favorable or unfavorable.

With much esteem,

Yours truly,

J. D. RUNKLE.

From the encouragement Mr. Runkle received, it would seem that the time had really come for the establishment of a mathematical journal which should become a permanent factor in directing the future development of mathematical thought in America.

The plan was well conceived, and such as to invite mathematical interests of every diversity.

In order to arouse interest among students of mathematics and reward patient industry, for here new discoveries are seldom found by accident, five problems were proposed entitled "Prize Problems for Students." A student in any Institution of Learning in the United States or British Provinces was eligible to compete for the prizes, which were two in number. The student sending the best solutions of the greatest number of prize problems in one number of the Journal in time for publication in the third number following the one in which the problems were proposed was given \$10; and the second in order of merit received a bound copy of the first volume of the *Monthly.*

To encourage research work, five prizes were offered for "essays upon any questions in pure or applied mathematics including those questions in physics, 'which can be solved only by an application of mathematical logic to the fundamental principles which constitute the scientific conception of the phenomena'."

> $50 was offered for the best essay; $40, for the next in order of merit; $30, for the third in order; $20 for the fourth in order; and $15 for the fifth in order.

The first number of the *Monthly* consisting of 28 pages, was published October, 1858.

Contents of Vol. I

Vol. I, No. I, October, 1858 contains Introductory note, page I; Circular note, page IV; names of contributors, page V; Announcement of prizes, pages XI,XII; Postscript to Introductory note, page 1; Prize Problems for students, page 2; and articles which will be listed later.

The most important article in the second number of Vol. I is, "An Account of the Comet of Donati," by G. P. Bond, Professor of Astronomy, Harvard University. This article which occupies 7 pages in the second number and 26 pages in the third, is embellished with beautiful steel engravings, which are some of the finest specimens of the steel-engraver's art. There are two beautiful full-page steel-engravings of the comet, one showing its appearance, October 2nd, 1858, and the other its appearance, October 10th, 1858.

The third number, Vol. I, contains an interesting "Note on the Theory of Probability," by Professor Simon Newcomb. Professor Newcomb proposed through the pages of the *Monthly* to develop the elementary principles of this Calculus and "to apply them to problems suggested either by the observation of the phenomena of the material world or by the affairs of society and daily life." Among other interesting problems discussed in these notes is Rev. Mr. Mitchell's problem concerning the probability that any six stars should fall in so small an area as that occupied by the Pleiades, if all the stars visible to the naked eye were scattered fortuitously over the heavens. Professor Newcomb defends Mr. Mitchell's solutions against the attack of the English scientist, Professsor Forbes.

Professor Newcomb's contributions to the *Mathematical Monthly* on the subject of Probability are contained in Nos. IV, VII, X, of Vol. I; Nos. IV and VIII, of Vol. II; and Nos. IV, and XI of Vol. III.

No. III also contains an article on the "Theory of the Distribution of Points on a line," by W. P. G. Bartlett.

No. IV, Vol. I, contains a demonstration of a "Prize Problem" by Professor Arthur Cayley. The problem, proposed by Professor H. A. Newton, of Yale University, reads as follows: "Four circles may be described, each of which shall touch the three sides of a plane triangle, or those sides produced. If six straight lines be drawn joining the centers of these circles, two and two, prove that the middle points of these six lines are in the circumference of a circle circumscribing the given triangles."

Instead of the four circles, Professor Cayley considers two of the circles, and the triangle as that formed by any three of the four common tangents of the two circles; the circle circumscribed about this triangle will pass through the middle point of the line of junction of the centers of the circles."

Considering the circle as a conic intersecting the line infinity in the circular points at infinity, the center of the circle is the pole of the line infinity with respect to the circle; the middle point of the line of junction of the two centers is the fourth harmonic with respect to the two centers and the point in which the line of junction meets the line infinity."

He then considers any two conics as intersecting in the points P, Q, the tangents at P, Q to the first conic intersecting in A and to the second conic in B, and let PQ meeting AB in O, and C be the fourth harmonic of O with respect to the points A and B. He then generalizes the theorem by substituting conics for circles and gives a demonstration by means of trilinear coordinates.

Professor W. Ferrell contributed to Nos. IV, VI, IX, XI, and XII, Vol. I and Nos. III, X, XI, Vol. II, an article of considerable interest on "Motions of Fluids and Solids Relative to the Earth's surface."

Rev. A. D. Wheeler, D. D., Brunswick, Me. contributed an article on "Indeterminate Analysis" in Nos. I, VI, and XII, Vol. II and on "Diophantine Analysis" in No. XI, Vol. III.

In No. II, Vol. II, is an article, "A complete list of the writngs of Nathaniel Bowditch," most of the "writings" being accompanied by a brief but interesting description, p. 57.

No. VII, Vol. III, contains an article on "The Theory of the Gyroscope" by Theodore G. Ellis, C. E., Boston; also a "complete catalogue of the writings of John Herschel, prepared by the author, p. 220.

No. XII of Vol. III, contains a "prize essay" entitled "Spherical Conics," by C. M. Woodward, Newburyport, Mass.

In this investigation, "the spherical conics" are the intersections made by a cone on the surface of a sphere whose center is at the vertex, the cone being any cone of the second degree.

The three volumes of this journal are full of just such material as the enthusiastic teacher of mathematics in High Schools, Academies, and Colleges is interested in and needs as supplementary material to arouse and maintain interest in his classes. No unfriendly criticism can be offered in reviewing this valuable publication. The beautiful diagrams, the fine typography, the careful selection and the good quality of the problems and articles, the eminent attainments of many of its contributors, and the painstaking care with which it was edited, all combined to make it satisfy most completely the requirements of a first class mathematical journal, and one well suited to satisfy the diversified interests of all classes of Mathematicians. Yet with all these excellent characteristics, it was, like all of its predecessors, forced to discontinue publication after only three volumes had been published.

Why should a journal so well suited to the needs of every teacher of mathematics be forced to suspend publication in so short a time? True, the Civil War was at hand and the minds of men were turned to the more pressing needs of the nation; yet it is very doubtful if such a number of the subscribers of the *Monthly* was actively engaged in that fierce struggle as to cause it to be discontinued through lack of support. The cause lies deeper. It seems to be due to a misconception entertained by a great majority of teachers of mathematics as to the necessary preparations requisite to the proper teaching of the subject. The notion is very common that any one capable of solving the problems in the ordinary arithmetics and algebras is an accomplished mathematician, while much less skill and knowledge is excusable on the part of one posing as a teacher of mathematics.

These self satisfied notions leave the teacher ignorant of the vast scope of mathematics and without ambition to pursue its study further. Thus, owing to the failure of many teachers of mathematics to recognize the duty they owe to themselves, the subject of mathematics, and their students, the editor of the *Mathematical Monthly* announced that one-third of the nearly eleven hundred subscribers to the first volume discontinued at the close.

He says, "I have supposed that those who continued their subscription to the second volume would not be so likely to discontinue it to the third volume, and I have made my arrangements accordingly. If, however, any considerable number should discontinue now, it will be subject to very serious loss. . . . I ask as a favor for all to continue to Volume III, and notify me during the year if they intend to discontinue at its close. I shall then know whether to begin the fourth volume. I shall not realize a dollar."*

*Quoted by Cajori, *The Teaching and History of Mathematics in the United States*, pp. 278, 279.

In the last number of Volume III, it is announced that "on account of the present disturbed state of public affairs, the publication of the *Mathematical Monthly* will be discontinued until further notice."* Thus, with No. 12, Vol. III, October, 1861, was discontinued the most promising mathematical journal which had, up to that time, appeared in America.

To the first volume, there were about 1100 subscribers. This number, instead of increasing, rapidly decreased until the end.

Among the contributors to its pages, were such prominent mathematicians as Professor Benjamin Peirce, Professor Theodore Strong, Professor G. P. Bond, and Professor W. G. Peck, and in addition a number of contributors who were just beginning their mathematical careers, some of whom have since achieved international reputations. Chief of these were Simon Newcomb, H. A. Newton, J. E. Oliver, Charles A. Young, J. M. Peirce, Artemas Martin, and De Volson Wood.

On page 32, Vol. II, among editorial items, the Editor announces, with great pleasure that the well-known and enterprising firm Ivison and Plinney have become the publishers of the *Mathematical Monthly* and that the firm has authorized him to say in advance of a fuller announcement that the *Mathematical Monthly* is now placed beyond any contingency and that it will continue to be published in Cambridge, (Mass.), under the supervision of the Editor and that its present character of being among the finest specimens of mathematical printing ever executed will be fully maintained.

*Cajori (l. c.).

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

The Conic Functions

BY ACHILLE CAPECELATRO
Fordham University and Brooklyn College

We are all familiar with the ordinary trigonometric functions, the sine and the cosine, and the manner in which they are related to the circle.

The circle is a special case of the conic section. Thus, are we not justified in asking this question, "Are there generalized *conic functions* of the *conic section* which will revert to the trigonometric functions when the conic section is a circle?" We shall proceed to answer this question as follows:

First we shall refresh our memories by deriving the equation of the conic section, and then showing the analytic method of deriving the ordinary trigonometric sine and cosine for the circle. By an analogous procedure, we shall derive the *conic functions*, and their properties, for the conic section.

I. The Equation of the Conic Section

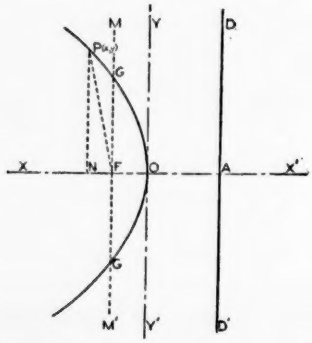
Let us define the *conic section* as the locus of a point moving in a plane so that the ratio of its distance from a fixed point, called the *focus*, to its distance from a fixed straight line, called the *directrix*, is constant.

This ratio we shall call the *eccentricity* of the conic section; and it will be denoted by e. It is evident that e can never be negative.

Let us construct the conic section from the definition. Let DAD', drawn vertically, be the directrix. Through A draw XAX' perpendicular to the directrix; we shall use XAX' as the $x-$axis. Mark a point F on the $x-$axis to the left of A; this point F will be the focus.

The conic section will intersect the $x-$axis at a point between F and A. This point of intersection we shall call O. Thus we have

$$FO = e \cdot OA. \tag{1.1}$$



We shall choose O as the origin of coordinates. And YOY' drawn perpendicular to XAX' at O shall be the $y-$axis.

At F construct MFM' perpendicular to the $x-$axis, and therefore parallel to DAD'. The conic section will intersect MFM' at G and G'. The distance $G'G$ is called the *latus rectum*. The distance

$$G'F = FG = \frac{G'G}{2}$$

is called the *semi-latus rectum* and it will be denoted by p; it is evident that p, as defined, cannot be negative. Therefore, by the definition,

$$p = e \cdot FA. \tag{1.2}$$

We may now proceed to obtain the equation of the conic section; using e and p as parameters. We should note that e is the parameter that governs the shape of the curve, and that p is the parameter that governs the size.

First let us obtain the value of the distance FO; from the diagram

$$OA = FA - FO.$$

Substituting this value of OA in (1.1)

$$FO = \frac{e \cdot FA}{1+e} \cdot$$

Substituting the value of FA from (1.2)

$$FO = \frac{p}{1+e} \cdot \tag{1.3}$$

Now let P be any point on the curve whose coordinates are

$$ON = x \quad \text{and} \quad NP = y. \tag{1.4}$$

Draw the focal radius FP. Then from the right triangle FNP

$$(NF)^2+(NP)^2=FP)^2. \tag{1.5}$$

But from the diagram

$$NF=NO-FO,$$

or, using (1.4) and (1.3),

$$NF=-x-\frac{p}{1+e} \cdot \tag{1.6}$$

And from the definition of the conic section

$$FP=e\cdot NA.$$

But $$NA= \quad NO+OA\,,$$

and, substituting from (1.4) and from (1.1) and (1.3),

$$NA=-x+\frac{p}{e(1+e)}\ ; \quad \text{and}$$

$$FP=-ex+\frac{p}{1+e} \cdot \tag{1.7}$$

Substituting (1.6), (1.4), and (1.7) in (1.5), we obtain

$$\left(-x-\frac{p}{1+e}\right)^2+y^2=\left(-ex+\frac{p}{1+e}\right)^2, \quad \text{or}$$

$$(1-e^2)x^2+y^2=-2px. \tag{1.8}$$

This last equation, (1.8), is true for all points on the conic section, and for no other point. It is therefore the *equation of the conic section in terms of the eccentricity e and the semi-latus rectum p.*

II. The Circle (Sine and Cosine)

The circle can be considered as the conic section with eccentricity zero.* Therefore, (1.8) with $e=0$ gives this equation for the circle:

$$x^2+y^2=-2px. \tag{2.1}$$

*From section I, $$e=\frac{FO}{OA} \;.$$

Thus e will equal zero if $FO=0$; this results in the trivial point conic. However, e can also equal zero by considering the limit of the above ratio for a finite FO and an infinitely increasing value of OA; which gives the circle with radius FO.

The formula for the differential arc of a curve is

$$ds = \sqrt{1 + \left(\frac{dx}{dy} \right)^2} dy .$$

Thus, substituting the value of $\frac{dx}{dy}$ from (2.1)

$$ds = \sqrt{1 + \frac{y^2}{p^2 - y^2}} \; dy, \quad \text{or}$$

$$ds = \frac{pdy}{\sqrt{p^2 - y^2}} . \tag{2.2}$$

The radius of the circle is equivalent in value to the semi-latus rectum p; so that if we divide ds by p, we have the differential value of the "*arc*" expressed in radians:

$$\frac{ds}{p} = \frac{dy}{\sqrt{p^2 - y^2}} . \tag{2.3}$$

Now let us integrate between the limits O and y and call the result Θ:

$$\Theta = \int_0^y \frac{ds}{p} = \int_0^y \frac{dy}{\sqrt{p^2 - y^2}} . \tag{2.4}$$

This is a well known integral, and we have

$$\Theta = \text{arcsine} \frac{y}{p} . \tag{2.5}$$

Which may be read as follows, Θ is the arc (variable) whose sine is

$$\frac{y}{p}, \quad \text{i. e., sine } \Theta = \frac{y}{p} .$$

Note that $$d\Theta = \frac{ds}{p} ;$$ and that we could

have written $$d\Theta = \frac{pds}{p^2} .$$

As p is the parameter for size, we can choose $p=1$ for convenience and now (2.4) is

dy

$$\Theta=\int_0^y \frac{dy}{\sqrt{1-y^2}}. \qquad \text{And} \tag{2.6}$$

$$\text{sine } \Theta = y. \tag{2.7}$$

We define:

$$\text{cosine } \Theta = \sqrt{1-y^2} = \sqrt{1-\text{sine}^2\Theta}. \tag{2.8}$$

By following these same steps with the generalized conic section, we shall proceed to derive the more general *conic functions*.

III. The Conic Functions

The equation of the conic section (1.8) is

$$(1-e^2)x^2+y^2=-2px. \tag{3.1}$$

By rectifying this equation we have:

$$ds=\sqrt{\frac{p^2+e^2y^2}{p^2+(e^2-1)y^2}}\,dy. \tag{3.2}$$

We may observe that $\sqrt{p^2+(e^2-1)y^2}$ is the expression for the length of the subnormal to the conic section; and that $\sqrt{p^2+e^2y^2}$ is the expression for the length of the normal to the conic section.

Now we shall choose the following variable v, (v *is to be the generalized "arc" or conic radian*):

$$d(v)=\frac{pds}{p^2+e^2y^2}. \tag{3.3}$$

Note: For the circle, $e=0$, and therefore (see note following (2.5).)

$$d(v)=\frac{pds}{p^2}=\frac{ds}{p}.$$

For convenience, we shall choose $p=1$; this will not alter our result at all, but will eliminate defining our functions as ratios with respect to p, i. e., for the circle we choose the radius (p) equal to unity so that the ordinate will equal the sine of the arc.

Integrating (3.3):

$$v=\int_0^v \frac{dy}{\sqrt{1+(e^2-1)y^2}\,\sqrt{1+e^2y^2}}\,. \tag{3.4}$$

Proceeding as in section II, we shall read (3.4) as follows:

This integral v is the variable whose conic sine is y. And we shall define these three conic functions:

$$\begin{aligned} &\operatorname{snc} v=y,\\ (3.5)\qquad &\operatorname{gnc} v=\sqrt{1+(e^2-1)y^2}=\sqrt{1+(e^2-1)\operatorname{snc}^2 v},\\ \text{and}\qquad &\operatorname{anc} v=\sqrt{1+e^2y^2}=\sqrt{1+e^2\operatorname{snc}^2 v}. \end{aligned}$$

From (3.4),

$$\frac{dv}{dy}=\frac{1}{\sqrt{1+(e^2-1)y^2}\,\sqrt{1+e^2y^2}}\,,$$

or

$$\frac{d\operatorname{snc} v}{dv}=\operatorname{gnc} v\cdot\operatorname{anc} v.$$

And by the defining equations (3.5):

$$\frac{d\operatorname{gnc} v}{dv}=(e^2-1)\operatorname{snc} v\cdot\operatorname{anc} v \tag{3.6}$$

and

$$\frac{d\operatorname{anc} v}{dv}=e^2\cdot\operatorname{snc} v\cdot\operatorname{gnc} v.$$

Utilizing the relationships given in (3.5) and (3.6), we derive these companion integrals to (3.4):

(3.7) $(e\neq 1)$, $\displaystyle\int\frac{dx}{\sqrt{x^2-1}\,\sqrt{e^2x^2-1}}$ equals the variable whose gnc is x.

If $e=1$, $\operatorname{gnc} v=1$.

(3.8) $(e\neq 0)$, $\displaystyle\int\frac{dz}{\sqrt{z^2-1}\,\sqrt{(e^2-1)z^2+1}}$ equals the variable whose anc is z.

If $e=0$, $\operatorname{anc} v=1$.

Let us examine the integrals (3.4), (3.7), and (3.8) for the following special values oJ the eccentricity e:

Case 1. $e=0$; the conic section is the circle.

$$(3.9)\qquad \begin{aligned} &(3.4)\rightarrow \quad \int \frac{dy}{\sqrt{1-y^2}} = \mathrm{snc}^{-1}\, y = \sin^{-1} y. \\ &(3.7)\rightarrow -\int \frac{dx}{\sqrt{1-x^2}} = \mathrm{gnc}^{-1}\, x = \cos^{-1} x. \\ &(3.8)\rightarrow \qquad \mathrm{anc}\, v = 1. \end{aligned}$$

Case 2. $e=1$; the conic section is the parabola.

$$(3.10)\qquad \begin{aligned} &(3.4)\rightarrow \quad \int \frac{dy}{\sqrt{1+y^2}} = \mathrm{snc}^{-1}\, y = \sinh^{-1} y. \\ &(3.7)\rightarrow \qquad \mathrm{gnc}\, v = 1. \\ &(3.8)\rightarrow \quad \int \frac{dz}{\sqrt{z^2-1}} = \mathrm{anc}^{-1}\, z = \cosh^{-1} z. \end{aligned}$$

From the defining equations of the conic functions, (3.5), we have

$$(3.11)\qquad \mathrm{snc}^2\, v + \mathrm{gnc}^2\, v = \mathrm{anc}^2\, v.$$

Geometrically, we have the normal as the hypotenuse of a right triangle which has the ordinate and the subnormal as the two sides. And again:

Case 1. $e=0$.

$$(3.11)\rightarrow \qquad \sin^2 v + \cos^2 v = 1.$$

Case 2. $e=1$.

$$(3.11)\rightarrow \qquad \sinh^2 v + 1 = \cosh^2 v.$$

Thus we find that we have derived, for the conic section, a set of *conic functions*; snc v, gnc v, and anc v. These *conic functions*, as we desired, become the trigonometric functions when the conic section is the circle. Also, they become the hyperbolic functions when the conic section is the parabola.

The student may be disturbed by the latter result, as he has been accustomed to associate the hyperbolic functions with the equilateral hyperbola. This change is a direct result of the choice which we have made for our variable. We may uphold our choice by reasoning as follows:

The circle can be considered as the ellipse with least eccentricity; and, for the circle, we find our conic functions reducing to the well known and simpler trigonometric functions, the sine and cosine.

Now the parabola is the hyperbola with least eccentricity; and, therefore, we should expect to find our conic functions reducing to the simpler hyperbolic functions, the sinh and cosh, for the parabola.

We should note that our conic functions may be related to the present elliptic functions as follows:

$$\operatorname{snc} v = \frac{\operatorname{sn} u}{\operatorname{dn} u}\,.$$

$$\operatorname{gnc} v = \frac{\operatorname{cn} u}{\operatorname{dn} u}\,.$$

$$\operatorname{anc} v = \frac{1}{\operatorname{dn} u}\,.$$

And if we let
$$\Phi = \tan^{-1} \frac{\operatorname{snc} v}{\operatorname{gnc} v}\,;$$

then Φ is the *amplitude* of u, and $\sin \Phi = \operatorname{sn} u$.

But we should like to observe that our conic functions are really the generalized trigonometric functions of the conic section, for they reduce directly to the trigonometric and hyperbolic functions for the correct values of the eccentricity. Also, that they were derived by a simple generalization of the analytic derivation of the trigonometric functions. Also, that after we found that the snc v, the gnc v, and the anc v were respectively the length of the ordinate, the length of the subnormal, and the length of the normal to the conic section. Then we had a simple geometric relationship for our conic functions. And this geometric relationship reduces to the well known circle diagram when, for $e=0$, our conic functions become the sine and cosine. (The anc is the radius.)

IV. Formulas and Properties of the Conic Functions

Restating equation (3.11),

$$\operatorname{snc}^2 v + \operatorname{gnc}^2 v = \operatorname{anc}^2 v. \tag{4.1}$$

From the defining equations, (3.5):

$$\operatorname{snc} v = \sqrt{\frac{\operatorname{gnc}^2 v - 1}{e^2 - 1}}\,, \tag{4.2}$$

$$\text{snc } v = \frac{\sqrt{\text{anc}^2 v - 1}}{e}, \tag{4.3}$$

$$\text{gnc } v = \sqrt{1 + (e^2-1)\text{snc}^2 v}, \tag{4.4}$$

$$\text{gnc } v = \frac{\sqrt{1+(e^2-1)\text{anc}^2 v}}{e}, \tag{4.5}$$

$$\text{anc } v = \sqrt{1+e^2 \text{ snc}^2 v}, \quad \text{and} \tag{4.6}$$

$$\text{anc } v = \sqrt{\frac{e^2 \text{ gnc}^2 v - 1}{e^2-1}}. \tag{4.7}$$

From (3.8), we also have by the use of Maclaurin's series:

$$\begin{aligned} &\text{snc}(0) = 0, \\ &\text{gnc}(0) = 1, \\ \text{and} \quad &\text{anc}(0) = 1. \end{aligned} \tag{4.8}$$

From (3.8), we also have by the use of Maclaurin's series:

$$\text{snc } v = v - (1-2e^2)\frac{v^3}{3!} + (1-16e^2+16e^4)\frac{v^5}{5!} - \cdots ; \tag{4.9}$$

$$\text{gnc } v = 1 - (1-e^2)\frac{v^2}{2!} + (1-6e^2+5e^4)\frac{v^4}{4!} - \cdots ; \quad \text{and} \tag{4.10}$$

$$\text{anc } v = 1 + e^2\frac{v^2}{2!} + (5e^4-4e^2)\frac{v^4}{4!} + \cdots . \tag{4.11}$$

By substitution in these series, or from the original defining equations, follow these formulas for the negative variable:

$$\begin{aligned} &\text{snc}(-v) = -\text{snc } v, \\ &\text{gnc}(-v) = \text{gnc } v, \\ \text{and} \quad &\text{anc}(-v) = \text{anc } v. \end{aligned} \tag{4.12}$$

By a familiar method,* we obtain the formulas for the conic functions of the sum (or difference) of two variables:

$$\text{snc}(v \pm w) = \frac{\text{snc } v \text{ gnc } w \text{ anc } w \pm \text{snc } w \text{ gnc } v \text{ anc } w}{1 - e^2(e^2-1)\text{snc}^2 v \text{ snc}^2 w}. \tag{4.13}$$

*Frederick S. Woods, *Advanced Calculus*, Ginn and Co., p. 372.

$$\text{(4.14)} \quad \operatorname{gnc}(v \pm w) = \frac{\operatorname{gnc} v \operatorname{gnc} w \pm (e^2-1)\operatorname{snc} v \operatorname{snc} w \operatorname{anc} v \operatorname{anc} w}{1-e^2(e^2-1)\operatorname{snc}^2 v \operatorname{snc}^2 w}.$$

$$\text{(4.15)} \quad \operatorname{anc}(v \pm w) = \frac{\operatorname{anc} v \operatorname{anc} w \pm e^2 \operatorname{snc} v \operatorname{snc} w \operatorname{gnc} v \operatorname{gnc} w}{1-e^2(e^2-1)\operatorname{snc}^2 v \operatorname{snc}^2 w}.$$

If we were to substitute first $e=0$, and then $e=1$, in these addition formulas, we would obtain the familiar addition formulas for the trigonometric and, in turn, the hyperbolic functions.

We can now find the periods of our conic functions. By considering the ellipse (a closed curve), we see that one fourth of the period shall have been attained when the length of the subnormal is zero and that then the length of the normal equals the length of the ordinate.

Thus, let us write

$$\operatorname{gnc} P = 0.$$

Then from (4.2) and (4.7),

$$\text{(4.16)} \qquad \operatorname{snc} P = \sqrt{\frac{1}{1-e^2}},$$

and

$$\operatorname{anc} P = \sqrt{\frac{1}{1-e^2}}.$$

As we have defined the conic functions as functions of a real variable (see the definition of the conic section, section I), we are limited to a consideration of real periods. Thus these last expressions, (4.16), are valid only for $e<1$.

From the addition formulas, let us find the values of the conic functions for $(v+P)$.

$$\operatorname{snc}(v+P) = \sqrt{\frac{1}{1-e^2}} \quad \frac{\operatorname{gnc} v}{\operatorname{anc} v}.$$

$$\text{(4.17)} \qquad \operatorname{gnc}(v+P) = -\frac{\operatorname{snc} v}{\operatorname{anc} v}.$$

$$\operatorname{anc}(v+P) = \sqrt{\frac{1}{1-e^2}} \quad \frac{1}{\operatorname{anc} v}.$$

Adding P to (4.17):

$$\begin{aligned} \operatorname{snc}(v+2P) &= -\operatorname{snc} v. \\ \operatorname{gnc}(v+2P) &= -\operatorname{gnc} v. \\ \operatorname{anc}(v+2P) &= \operatorname{anc} v. \end{aligned} \tag{4.18}$$

And now adding $2P$ to (4.18),

$$\begin{aligned} \operatorname{snc}(v+4P) &= \operatorname{snc} v. \\ \operatorname{gnc}(v+4P) &= \operatorname{gnc} v. \\ \operatorname{anc}(v+4P) &= \operatorname{anc} v. \end{aligned} \tag{4.19}$$

Although imaginary periods do not exist in the present domain of definition of our functions, we know that by an extension to the complex variable these non-existent periods would become very actual. Being careful to keep this in mind, let us try also

$$\operatorname{anc} P' = 0.$$

Then from (4.3) and (4.5),

$$\operatorname{snc} P' = \frac{i}{e}, \tag{4.20}$$

and

$$\operatorname{gnc} P' = \frac{1}{e}.$$

We may then state that

snc v has the two periods $4P$ and $4P'$,

gnc v has the two periods $4P$ and $2P'$, and

anc v has the two periods $2P$ and $4P'$.

With these definitions for P and P':

$$P = \operatorname{snc}^{-1} \sqrt{\frac{1}{1-e^2}},$$

and

$$P' = \operatorname{snc}^{-1} \frac{i}{e}.$$

Let us find the periods of the conic functions for our two special values of the eccentricity:

Case 1. $e=0$.

$$\text{snc } P=1=\sin P.$$

Therefore, in this case

$$P=\frac{\pi}{2} \quad \text{and} \quad 4P=2\pi$$

which is correct for the trigonometric functions.

$$P' \text{ does not exist for } e=0.$$

Case 2. $e=1$. (Recalling our remarks on imaginary periods.)

$$P \text{ does not exist for } e=1.$$

$$\text{snc } P'=i=\sinh P'.$$

But $\sinh P'=-i \sin iP'$; therefore $\sin iP'=-1$, $ip'=-\pi/2$, and $4P'=2\pi i$ which is correct for the hyperbolic functions.

We have called the snc v the conic sine of v. The gnc v and the anc v have been left unnamed.

Mathematical World News

Edited by
L. J. Adams

Dr. Ellis B. Stouffer has been appointed head of the mathematics department of the University of Kansas by Chancellor Deane W. Malott. He succeeds Dr. U. C. Mitchell, whose resignation was accepted January 7, 1941. Dr. Stouffer holds two positions at the university, being dean of the graduate school and chairman of the university budget committee. Dr. Mitchell will be on sabbatical leave during the second semester of the current school year.

The Southern California Section of the Mathematical Association of America is scheduled to meet at the University of Redlands in Redlands, California on March 8, 1941. Professor O. W. Albert is chairman of the Section. The program committee consists of Professor Paul H. Daus of the University of California at Los Angeles, Professor D. V. Steed of the University of Southern California, and Professor Hugh Hamilton of Pomona College.

The Proceedings of the Industrial Safety Conference of November 8, 1940 is a bulletin of the Virginia Polytechnic Institute. In it are copies of addresses dealing with matters of industrial safety from an engineering standpoint.

Mr. Howard B. Kingsbury, head of the mathematics department at West Division High School of Milwaukee, Wisconsin was killed in a traffic accident on January 17, 1941. A graduate of the University of Illinois, Mr. Kingsbury had been a teacher at West Division High School for twenty-eight years.

In the January 18 issue of *School and Society*, Professor B. A. Miller, University of Illinois, discusses the difference between *arithmetic* and *algebra.*

The following report on the annual meetings of the Mathematical Association of America, the American Mathematical Society, and the National Council of Teachers of Mathematics is presented through the courtesy of Professor C. D. Smith, Mississippi State College: The annual meetings of the American Mathematical Society, the Mathematical Association of America, and the National Council of Teachers of Mathematics, were held at the Louisiana State University, December

30, 1940 to January 2, 1941. Professor S. T. Sanders reported that the local committee on arrangements was pleased with the large attendance of visitors from other sections. The attendance from Louisiana and adjacent states was representative of all mathematical interests including large delegations from high schools and junior colleges. The buildings and facilities of the University were adequate for all purposes and the local committee spared no pains to see that visitors were provided with every convenience and comfort. Special reports on the plans of committees for mathematics in the national defense program, meetings of the staffs of the mathematical publications, and meetings of other important committees were high lights of the program of business. Conversations about the hotel lobbies reflected general opinion that education faces a crisis along with our present program of preparedness, and we must carry on this year with increased vigilance and vigor. Some one said that it seems that "We must be faithful or fail; we must be loyal or we will be lost."

Professor William Caspar Graustein, professor of mathematics and assistant dean at Harvard University, was killed in an automobile accident on January 22, 1941.

Professor Marston Morse, Institute for Advanced Study, Princeton, New Jersey, was elected President of the American Mathematical Society at the meeting of the Society in Baton Rouge, Louisiana. Professor T. Y. Thomas, University of California at Los Angeles, was elected Vice-President. Professor J. R. Kline, University of Pennsylvania, was elected Secretary.

Professor F. R. Moulton, permanent secretary of the American Association for the Advancement of Science, delivered an address on *Mathematics as Related to Astronomy* at Rutgers University. The occasion was the 175th anniversary convocation of the University.

The National Council of Teachers of Mathematics was scheduled to meet at the Hotel Chelsea, Atlantic City, New Jersey on February 20-22. The theme for the meeting was *Mathematics in a Defense Program.*

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 132. (Corrected.) Proposed by *William E. Byrne*, V. M. I.

Let (x_0, y_0) be a point P_0 of the curve C, $y=f(x)$. Under the assumptions that $f(x)$ admits a Taylor development about the point P_0 and that $f''(x_0), f'''(x_0) \neq 0$, find

$$\operatorname*{Limit}_{x_1 \to x_0} \frac{(x_1-x_0)^n+(x_2-x_0)^n}{(x_1+x_2-2x_0)}, \qquad (n=2, 3, \cdots)$$

where $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are points of C and the chord P_1P_2 is parallel to the tangent P_0T_0 at P_0. What happens if $f''(x_0)=0$, $f'''(x_0)=0$?

Solution by the *Proposer*.

It is convenient to take P_0 as the origin of coordinates by making a preliminary translation. Hence the problem is to determine

$$\lim_{x_1 \to 0} \frac{x_1{}^n+x_2{}^n}{x_1+x_2}, \quad (n=2, 3, \cdots).$$

By hypothesis

$$f'(0)=(y_2-y_1)/(x_2-x_1)$$

$$=\frac{f'(0)(x_2-x_1)+\frac{1}{2}f''(0)(x_2{}^2-x_1{}^2)+f'''(0)(x_2{}^3-x_1{}^3)/6+\cdots}{x_2-x_1}$$

$$=f'(0)+\tfrac{1}{2}f''(0)(x_2+x_1)+f'''(0)(x_2{}^2+x_2x_1+x_1{}^2)/6+\cdots.$$

Thus x_2 is given as an implicit function of x_1 by the relation

$$(1) \quad F(x_1,x_2) \equiv f''(0)(x_2+x_1)/2!+f'''(0)(x_2{}^2+x_2x_1+x_1{}^2)/3!+\cdots=0.$$

Assume $\partial F/\partial x_2 \neq 0$* for the entire neighborhood to be considered. There then exists a single-valued, continuous function

$$x_2 = \varphi(x_1), \quad \varphi(0) = 0$$

having a limited Taylor development about $x_1 = 0$ and such that

$$F[x_1, \varphi(x_1)] = 0.$$

Thus we may take

$$x_2 = -x_1 + a_2 x_1^2 + a_3 x_1^3 + \cdots$$

which gives, upon substitution in (1),

$$\begin{aligned} (2) \qquad & f''(0)[a_2x_1^2 + a_3x_1^3 + a_4x_1^4 + \cdots]/2! \\ & + f'''(0)[x_1^2 - a_2x_1^3 + (a_2^2 - a_3)x_1^4 + \cdots]/3! \\ & + f^{iv}(0)[2a_2x_1^4 + \cdots]/4! + f^{v}(0)[x_1^4 + \cdots]/5! + \cdots = 0. \end{aligned}$$

This gives at once

$$a_2 = -f'''(0)/3f''(0).$$

The following computations are obvious:

$$\frac{x_1^n + x_2^n}{x_1 + x_2} = \frac{2x_1^n + \cdots}{a_2x_1^2 + \cdots} \quad \text{for } n \text{ even},$$

$$\frac{x_1^n + x_2^n}{x_1 + x_2} = \frac{na_2x_1^{n+1} + \cdots}{a_2x_1^2 + \cdots} \quad \text{for } n \text{ odd}.$$

Hence
$$\frac{x_1^n + x_2^n}{x_1 + x_2} = \frac{2}{a_2} = \frac{-6f''(0)}{f'''(0)} \quad \text{for } n = 2$$

$$= 0 \quad \text{for } n = 3, 4, \cdots; \quad f''(0), \quad f'''(0) \neq 0.$$

If $f''(0) = 0$, $f'''(0) \neq 0$, then the curve C has an inflection point at P_0 and it is geometrically evident that no real solution

$$x_2 = \varphi(x_1), \quad \varphi(0) = 0$$

exists.

For $f''(0) \neq 0$, $f'''(0) = 0$, $f^{iv}(0) \neq 0$, equation (2) gives at once

$$a_2 = 0, \quad a_3 = 0, \quad a_4 = -f^{v}(0)/60f''(0),$$

*The condition $\partial F/\partial x_2 = 0$ implies that, with P_1 fixed and while P_1P_2 remains parallel to P_0T_0, x_2 can undergo an infinitesimal change. Thus P_1P_2 is also tangent to C at P_2. If there exists such a singular point between P_2 and P_0, then, as $P_1 \to P_0$, P_2 will not approach P_0 but will instead approach some point T' at which P_0T_0 again meets the curve C. In such a case the desired limit is evidently X^{n-1} where X is the abscissa of T'. When $f''(0) \neq 0$, we always have $\partial F/\partial x_2 \neq 0$ for some neighborhood of P_0.

from which we calculate, as in the original case,

$$\frac{x_1^n+x_2^n}{x_1+x_2}=\frac{2x_1^n+\cdots}{a_4x_1^4+\cdots}\text{ for } n \text{ even,}$$

$$\frac{x_1^n+x_2^n}{x_1+x_2}=\frac{na_4x_1^{n+3}+\cdots}{a_4x_1^4+\cdots}\text{ for } n \text{ odd.}$$

Thus the desired limit is ∞ for $n=2$, 0 for $n=3$, $2/a_4=-120f''(0)/f^{v}(0)$ for $n=4$, and 0 for $n=5, 6, \cdots$.

Similarly for $f''(0)=0$, $f'''(0)=0$, $f^{iv}(0)\neq 0$, $f^{v}(0)\neq 0$ we find a real solution

$$x_2=\varphi(x_1)=-x_1+a_2x_1^2+\cdots$$

in which $a_2=-f^{v}(0)/10f^{iv}(0)$. Thus the desired limit is

$$-20f^{iv}(0)/f^{v}(0) \text{ for } n=2, \text{ and } 0 \text{ for } n=3, 4, \cdots .$$

The limit is needed in solving the following problem: to determine the slope of the segment P_0P_3, where P_3 is the midpoint of P_1P_2 given above.

No. 352. Proposed by *Howard D. Grossman*, New York, N. Y.

If each side of a triangle be divided into n equal parts and through the points of division lines be drawn inside the triangle parallel to each side, then the total number of triangles in the whole figure is the integer nearest to $n(n+2)(2n+1)/8$.

Solution by the *Editors*.

Consider a particular side BC of the original triangle ABC as base and let BC/n be the unit of length. Let $B'C'$, one of the lines parallel to BC, be k units long. If $k\leqq\frac{1}{2}n$, $B'C'$ serves as base of k small triangles whose base is 1 unit, $(k-1)$ triangles of base $2, \cdots, 1$ triangle of base k, on either side of $B'C'$. Thus there are

$$(1)\qquad 2(1+2+3+\cdots+k)=k(k+1)$$

triangles based on $B'C'$. If however $k>\frac{1}{2}n$, the largest triangle with base on $B'C'$ and lying opposite A has a base $(n-k)$ units long. Hence the total given in (1) must be reduced by $(2k-n)$ triangles of base $(n-k+1)$, $(2k-n-1)$ of base $(n-k+2), \cdots$, 1 of base k, all of which would extend outside ABC. Since

$$(2k-n)+(2k-n-1)+\cdots+1=\tfrac{1}{2}(2k-n)(2k-n+1),$$

the grand total of all triangles is

$$(2) \qquad \sum_{k=1}^{n} k(k+1) - \sum_{k=|n/2|+1}^{n} \tfrac{1}{2}(2k-n)(2k-n+1).$$

The first term of (2) is easily reduced to $n(n+1)(n+2)/3$. If n is even, $2k-n=2u$ transforms the second term of (2) into

$$\sum_{u=1}^{n/2} u(2u+1) = n(n+2)(2n+5)/24$$

whence (2) becomes

$$n(n+2)(2n+1)/8.$$

If n is odd, $2k-n+1=2u$ transforms the second term of (2) into

$$\sum_{u=1}^{(n+1)/2} u(2u-1) = (n+1)(n+3)(2n+1)/24$$

whence (2) becomes

$$[n(n+2)(2n+1)-1]/8$$

which is clearly the integer nearest $n(n+2)(2n+1)/8$.

No. 373. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Find the smallest possible base of a system of numeration in which the three-digit number 777 is a perfect fourth power.

Solution by *Gertrude S. Ketchum*, Urbana, Illinois.

If b is the base and n is an integer such that $n^4=777$, then

$$n^4 = 7(b^2+b+1).$$

Hence 7 is a factor of n. With $n=7q$ we have

$$343q^4 = b^2+b+1.$$

The value $q=1$ gives $b=18$, which is obviously the smallest base. All values of b are given by the expression

$$b = \tfrac{1}{2}(-1 \pm \sqrt{1372q^4-3}).$$

Therefore, the possible integral values of b would be infrequent. By actual trial it is found that $q=2, 3, \cdots, 12$ do not give rational values for b. Hence 18 is the only value less than 3000.

Also solved by the *Proposer*.

No. 374. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Let Q be the foot of the perpendicular from the point P upon the polar of P with respect to the conic $Ax^2+By^2=C$.

(1) If P describes a straight line, then Q describes in general a cubic curve.

(2) If P describes a diameter of the conic, then Q describes an equilateral hyperbola.

(3) If P describes a line parallel to an axis of the given conic, then Q traces a circle with center on the other axis.

Solution by *H. T. R. Aude*, Colgate University.

Let (α,β) be the coordinates of the point P. Then the equation of the polar of P with respect to the conic $Ax^2+By^2=C$ is

$$A\alpha x+B\beta y=C. \tag{1}$$

The perpendicular to this polar through the point P has the equation

$$B\beta x-A\alpha y=(B-A)\alpha\beta, \tag{2}$$

and the point Q with coordinates (x, y) is the intersection of the lines (1) and (2).

(1) If P moves on the straight line $y=mx+b$, then

$$\beta=m\alpha+b. \tag{3}$$

Using this value of β in equations (1) and (2) gives respectively the equations

$$\alpha(Ax+Bmy)=C-Bby, \quad \text{and} \tag{4}$$

$$m(A-B)\alpha^2+(Bmx-Ay+Ab-Bb)\alpha+Bbx=0. \tag{5}$$

Elimination of α between these will show that in general there results an equation of the third degree. It should be noted, however, that if $A=B$ the resulting equation is

$$Bb(x^2+y^2)+(Cm-Bbm)x+(Bbm-C)y=0,$$

which represents a circle or a straight line. In the following, assume $A\neq B$.

(2) Let the point P move on any diameter of the conic except those given by $x=0$ and $y=0$. On account of equation (3) we have $b=0$, and there results from (4) and (5) the equation

$$(Ax+Bmy)(Bmx-Ay)=C(B-A)m.$$

This represents an equilateral hyperbola.

(3) Finally, if the point P moves on a line parallel to the x-axis, then m is equal to zero. Equations (4) and (5) yield the relation

$$ABb(x^2+y^2)+(B^2b^2-ABb^2-AC)y+(A-B)Cb=0.$$

This represents a circle with center on the y-axis. If P moves on the x-axis, then the locus of Q is the x-axis.

On account of symmetry corresponding loci will result when the point P moves on a line parallel to the y-axis or on the y-axis.

Also solved by *W. T. Short* and the *Proposer.*

No. 376. Proposed by *Walter B. Clarke*, San Jose, California.

The incircle of triangle ABC touches its sides at A_1, B_1, C_1. A_1B_1 is cut by the bisector of angle A at C_a, by the bisector of angle B at C_b; B_1C_1 is cut by the bisector of angle B at A_b, by the bisector of angle C at A_c; C_1A_1 is cut by the bisector of angle C at B_c, by the bisector of angle A at B_a. Show that A_bB_a, A_cC_a, and B_cC_b form the medial triangle of ABC.

Solution by the *Proposer.*

We have directly: $\angle CA_1B_1=90-C/2$; $\angle BA_1B_1=90+C/2$. Thus $\angle A_1C_bB=A/2$ and quadrilaterals BC_aC_bA and IB_1C_bA are each cyclic (I is the incenter of ABC). Since $\angle IB_1A$ is a right angle, so also is $\angle BC_bA$. Thus the center of the circle circumscribing quadrilateral BC_aC_bA is at the midpoint of AB.

In triangle $A_1B_cC_b$, B_cI is perpendicular to A_1C_b and C_bI is perpendicular to A_1B_c. Thus I is the orthocenter of $A_1B_cC_b$ and B_cC_b is perpendicular to A_1I. Hence B_cC_b is parallel to BC.

Let B_cC_b cut AB at C_m. Now $\angle C_bC_mA=\angle B=2(\angle C_bBA)$ and accordingly C_m is the center of the circle mentioned; i. e., the midpoint of AB. This establishes the line B_cC_b as the bisector of AC. Similarly for A_bB_a and A_cC_a.

Also solved by *D. L. MacKay.*

PROPOSALS

No. 397. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Find the envelope of all spheres which are bisected by two given intersecting spheres.

No. 398. Proposed by *John H. Giese*, New Brunswick, N. J.

An elliptic billiard table is assumed to be frictionless and to have perfectly elastic cushions. It has one pocket, at a vertex. Show that, if a ball placed at a focus be struck in any direction, it is sure to drop into the pocket.

No. 399. Proposed by *Nelson Robinson*, Louisiana State University.

A and B are playing a game at a round table using an unlimited number of flat discs having equal radii. Each player, in turn, places a disc flat against the top of the table. After a disc is once placed on the table, its position cannot be altered. The winner is the player who places a disc in the last available space on the table. In order to win, should A move first or second and what should be his method of play?

No. 400. Proposed by *Howard D. Grossman*, New York City.

Prove that if a coin is tossed often enough, the chance approaches 1 that the number of heads will eventually at least once exceed the number of tails by 1. Does this prove that if you play long enough, you are practically certain to win (or equally certain to lose)?

No. 401. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given angle C and side c, construct the scalene triangle ABC so that two external angle bisectors are equal.

No. 402. Proposed by *E. P. Starke*, Rutgers University.

Three integral squares are to be in harmonic progression: find the smallest set. What is the general solution?

No. 403. Proposed by *W. Raymond Crosier*, student, Colgate University.

Consider the one-parameter family of curves given by the equation

$$y=x^3-3x^2+kx+k,$$

where the parameter k is restricted so that there exist points of maxima and minima. Find the equation of the locus of these points.

No. 404. Proposed by *E. P. Starke*, Rutgers University.

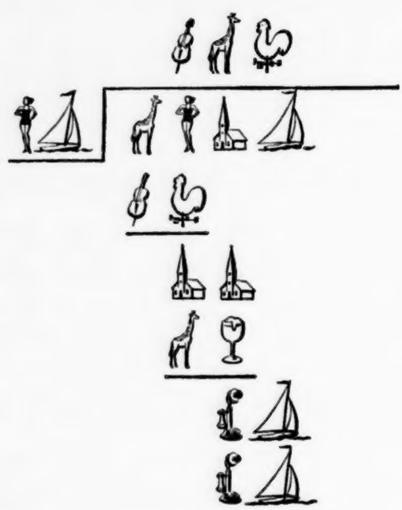
Compute $$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

No. 405. Proposed by *Robert C. Yates*, Louisiana State University.

Given the three distinct non-collinear points A, B, C. With the compasses alone, draw the circle inscribed to triangle ABC.

No. 406. Proposed by *Comptometer*, Chicago.

Identify the symbols in the accompanying long division example.



Bibliography and Reviews

Edited by
H. A. SIMMONS and JOHN W. CELL

The Human Worth of Rigorous Thinking (3rd ed.) By Cassius J. Keyser Published by Scripta Mathematica, Yeshiva College, N. Y. 1940.

The articles in this collection first appeared in various forms one to two score years ago. Their first appearance in book form occurred, for most of them, in 1916. The third edition of this book, made necessary by continuing demands, is now in the hands of reviewers. Inasmuch as the editions of the book have differed from each other very little, and have during the past twenty-five years already twice been thoroughly reviewed, it would seem useless indeed to do that job again. Upon this page in particular no effort will be made to do it. But to say that Professor Keyer's famous essays do not deserve a third edition of reviews, is not to say that they deserve to be neglected. The fact that circumstances have persuaded its publishers to issue the book again means that it is not being neglected by readers, and an inquiry into the reasons for their interest ought to be a more profitable pursuit than reviewing.

One good reason why the book still lives is easy to find. Many of the wind-mills with which Professor Keyer fought duels thirty years ago are still thrashing the air. There are, for example, still many fluent people who do not like mathematics, who think that it has no human significance, who believe that instruction in it must be necessarily cut and dried, and who consider that all rigorous thinking is without value anyway. Against such as these Professor Keyser's armied arguments, scholarly quotations, and persuasive prose are just as appropriate as ever. To those who sympathize with him it is disconcerting, in a way, to find that this is so. His efforts are well directed now because they have failed in their effect and because the battle lines have not changed.

This is the case in other respects also. Concerning mathematical productivity in the United States and the handicaps under which it struggles, Professor Keyser would find no reason to modify his comments. Regarding the permanent bases of liberal education in the United States no new bulletins of significance can be required of him. And yet there is one interesting instance in which this is not the case. No longer need "students of maturity and power who, whilst specializing in one subject or one field, desire to generalize" in mathematics, look in vain for courses suitably untechnical for their shorn condition. Orientation and survey courses are common now. It would be a treat to hear from Professor Keyser, even in his guarded prose, just what he thinks of them.

Although the topics which he chooses have stood the test of time very well, the same cannot be said of the style in which he expresses them. The past quarter century has made the invited lecturer polishing phrases in the center of a stage too large for his talents, look inexpressibly old fashioned. Today is a day of directness and brevity; news flashes, business surveys, graphs, indexes, and fireside chats. Time presses; but not Professor Keyser. He is the accomplished lecturer at the center of a large stage. He, beyond a doubt, polishes phrases. If you are a bit of an antiquary you may appreciate them.

Louisiana State University. N. E. RUTT.

Trigonometry (with Tables). By Ernst R. Breslich and Charles A. Stone. Laidlaw Brothers, New York, 1940. viii+180 pages.

The author's statement that trigonometry is a true correlation of arithmetic, algebra, and intuitive geometry is well borne out in the treatment of the subject. The close interrelaton of the topics presented is brought out in a way which is usually lacking in a trigonometry text. The chapter headings are: Trigonometric Ratios, Trigonometric Functions of Angles of any Size, Logarithms, The Slide Rule, The Solution of Oblique Triangles, Relations of the Trigonometric Functions, and Supplementary Topics. The last chapter includes a treatment of Complex Numbers, De Moivre's Formula, and Trigonometric and Exponential Series

From the beginning, the inductive method of approach is used. Concrete situations are presented, and the trigonometric ratios are used before a formal definition is set up. The transition from the simplest to more complex situations is made by easy, normal procedures. The authors draw freely upon algebra and geometry, and review such portions of those subjects as are necessary at the time. The proofs of the formulas are direct and compact. A neat geometric proof of the law of tangents is given. Identities, inverse functions and trigonometric equations are postponed to Chapter VI.

Among the best features as to content are the abundant and well-graded exercises which take care of the problem of individual differences. There is a clear presentation of the slide rule and its uses, and short historical sketches are introduced at appropriate times.

The book is attractive in appearance, written on good paper, and contains a good selection of examples for the aid of the student. There are complete tables in the back of the book, but it does not contain answers to any of the problems.

The extension of the meaning of the trigonometric functions for angles of all magnitudes and the reduction formulas for those angles are made to seem logical and natural. Although the text is somewhat longer than is usual, the material is well arranged and presented. The book should be popular for its teachability.

Louisiana State University. R. L. O'QUINN.

The Training of Mathematics Teachers for Secondary Schools in England and Wales and in the United States. By Ivan Stewart Turner. Bureau of Publications, Teachers College, Columbia University, New York, 1939. xiii+231 pages. $1.25.

This book, published as the Fourteenth Yearbook of the National Council of Teachers of Mathematics under the editorship of Professor William D. Reeve of Columbia University, presents in detail a study by Dr. Turner of the training of mathematics teachers in England and Wales and in the United States.

Dr. Turner assumes that persons chosen as prospective teachers possess certain qualifications of personality and character, broad cultural education, and richness of experience. He reviews some previous studies dealing with certain aspects of the training of mathematics teachers in England and Wales and in the United States and finds a number of relevant questions unanswered. He proposes seven to be considered in his study that are concerned with the amount of mathematics studied and the standards achieved in the two countries, the academic training of mathematics teachers adequate to guarantee reasonably good teaching, the nature and effectiveness of the present methods of professional training of mathematics teachers, the provision for in-service training, the nature and importance of the contribution of agencies other than the regular teacher training institutions, the assumptions implicit in the methods

of training adopted, and the principles on which the methods of training should be based. Nine principles fundamental to the training of mathematics teachers are defined to provide criteria in terms of which the methods of training can be evaluated.

The presentation includes analyses of secondary education, mathematics in the secondary schools, academic preparation of mathematics teachers, and professional training of prospective mathematics teachers. In each case data are presented for England and Wales and for the United States and comparisons are made. A final evaluation is offered in terms of the nine principles originally defined.

It is revealed among other interesting comparisons and conclusions that pupils in England and Wales are required to study considerably more mathematics and to reach higher standards somewhat earlier than pupils of a corresponding age in the United States; that a higher percentage of mathematics teachers in secondary schools in England and Wales are qualified to teach all the mathematics offered in those schools than is the case in the United States.

The book is recommended for its valuable content and attractive presentation to all those interested in the teaching of secondary school mathematics.

State Teachers College, DeKalb, Ill. EUGENE W. HELLMICH.